Credit Markets and Modeling

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Cognitive Capital

Debts and Default

- Bonds, Loans and Other Mischief
- Default
  - Capital Structure
  - Negotiation
  - Recovery

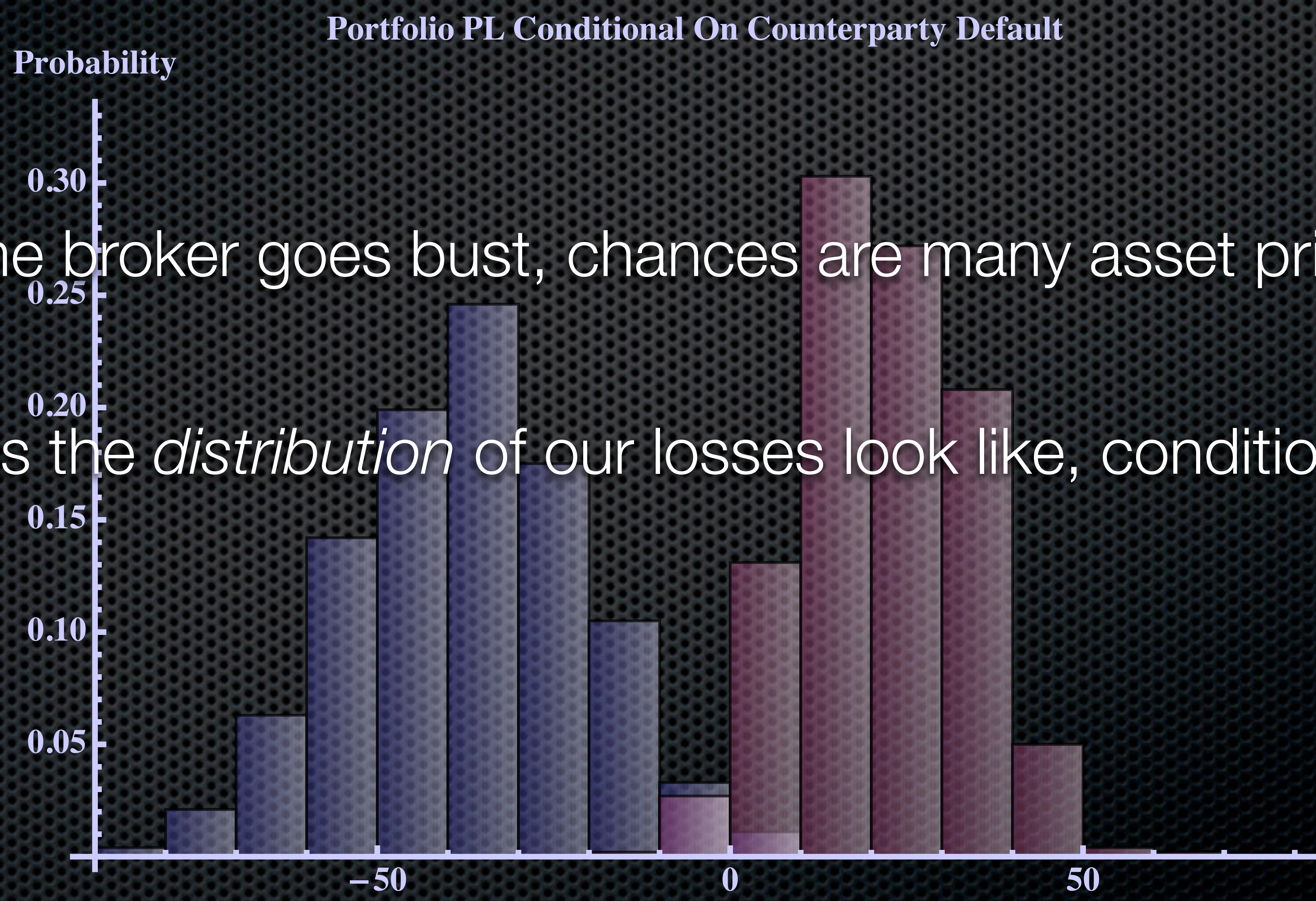
Promises, Promises, Promises

- Counterparty Risk
  - Accounts
  - Contracts
- Debt Instruments
  - Loans, Bonds, CDS, Converts, Prefs
  - Collateralized Obligations

Counterparty Risk

- Account risk
  - Lehman, MF Global
  - Margin accounts
  - Rehypothecation
- Contract Risk: OTC Derivatives

Counterparty Risk (Conditional Losses)

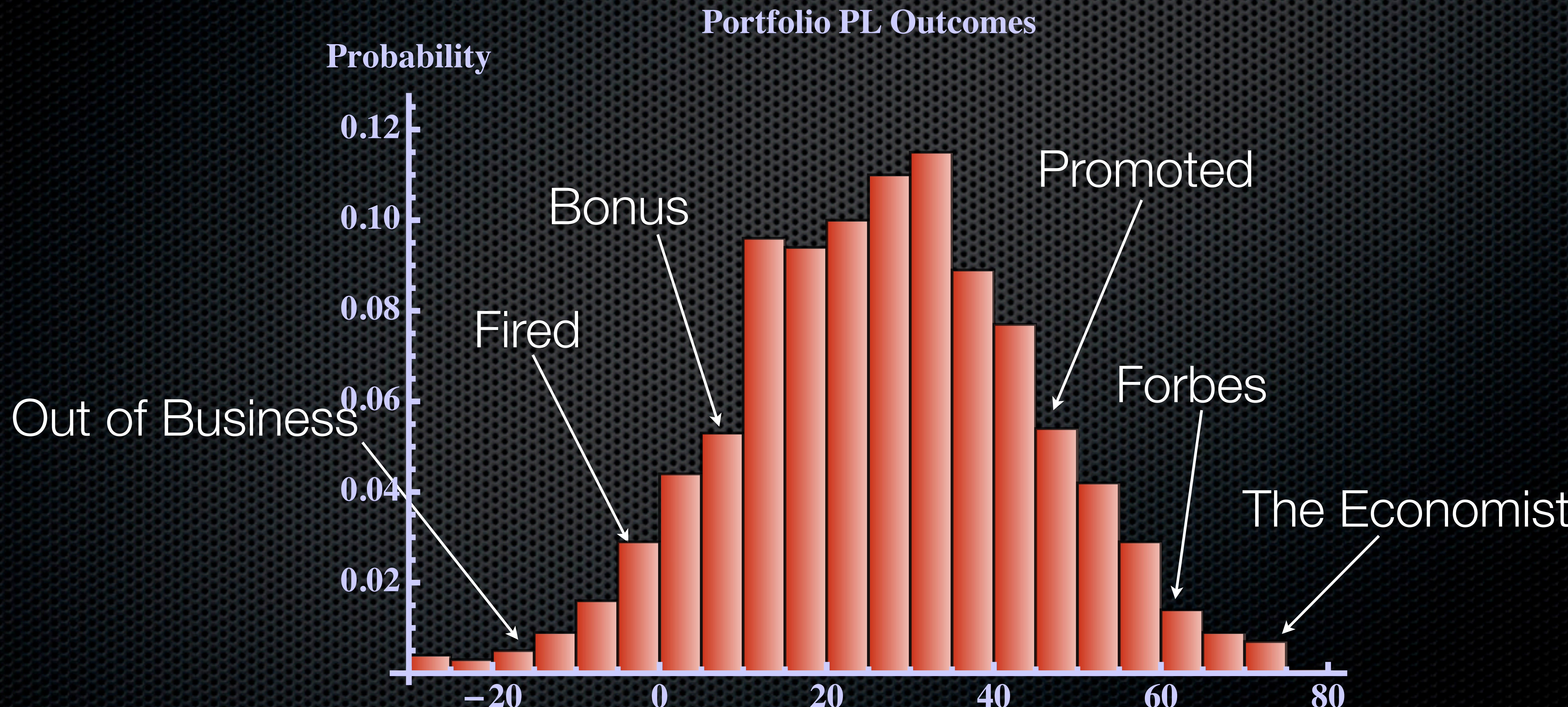


- If our prime broker goes bust, chances are many asset prices have dropped
- What does the *distribution* of our losses look like, conditional on that default?

Portfolios and Losses

- In a big portfolio, some losses are inevitable
- We can afford to be less concerned about any individual outcome

Portfolios, Profits and Losses



What Affects Credit Risk?

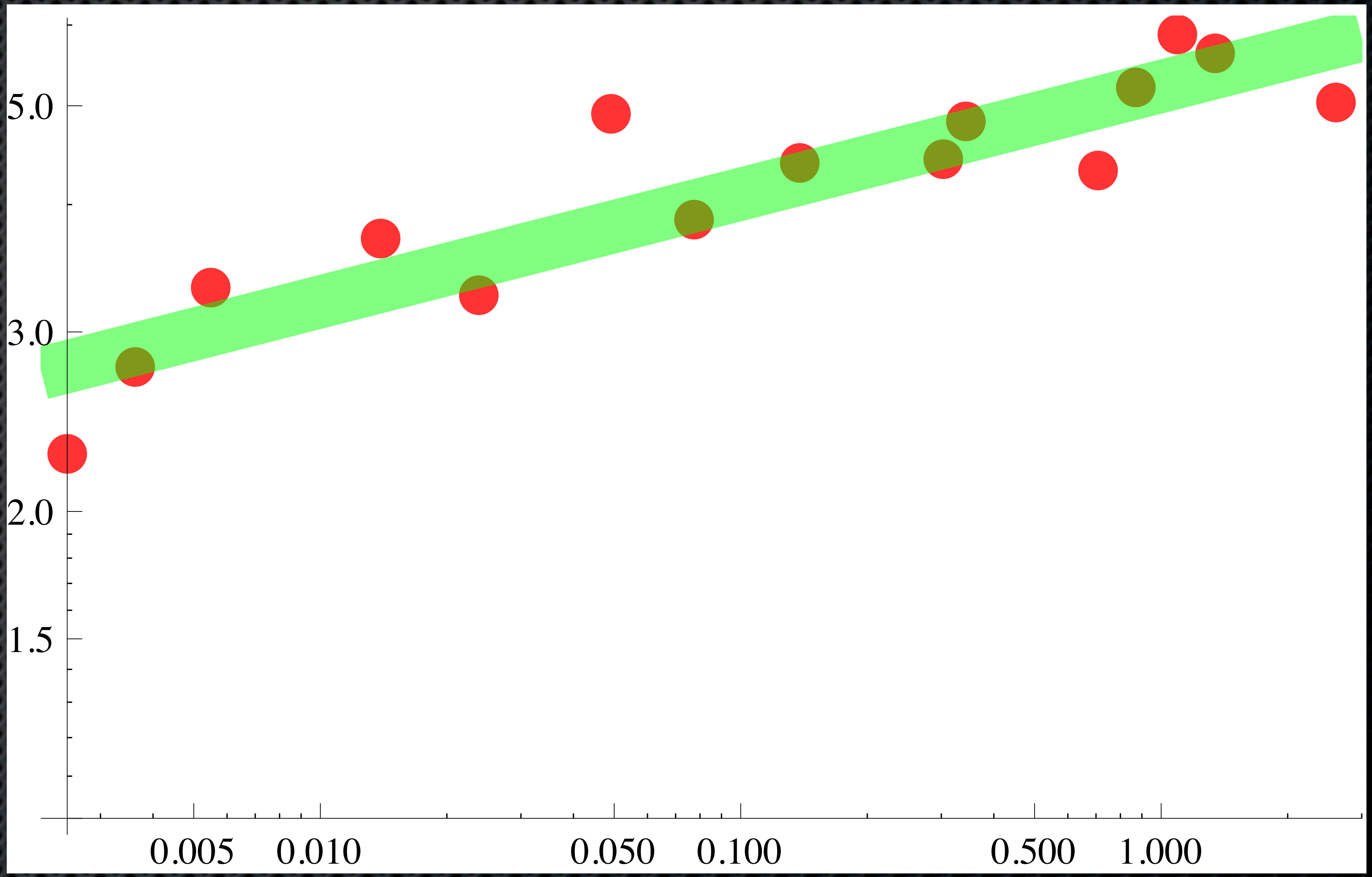
- Leverage/Capital Structure
- Volatility in Profits
- Refinancing

Agents of Change

- Business Decline
- Economic Stress
- Fraud
- Capital Structure Changes

Complexity = Simplicity

- Complex drivers
- Trying to calibrate a model to something that has never happened before



Complexity = Simplicity

- Complex drivers
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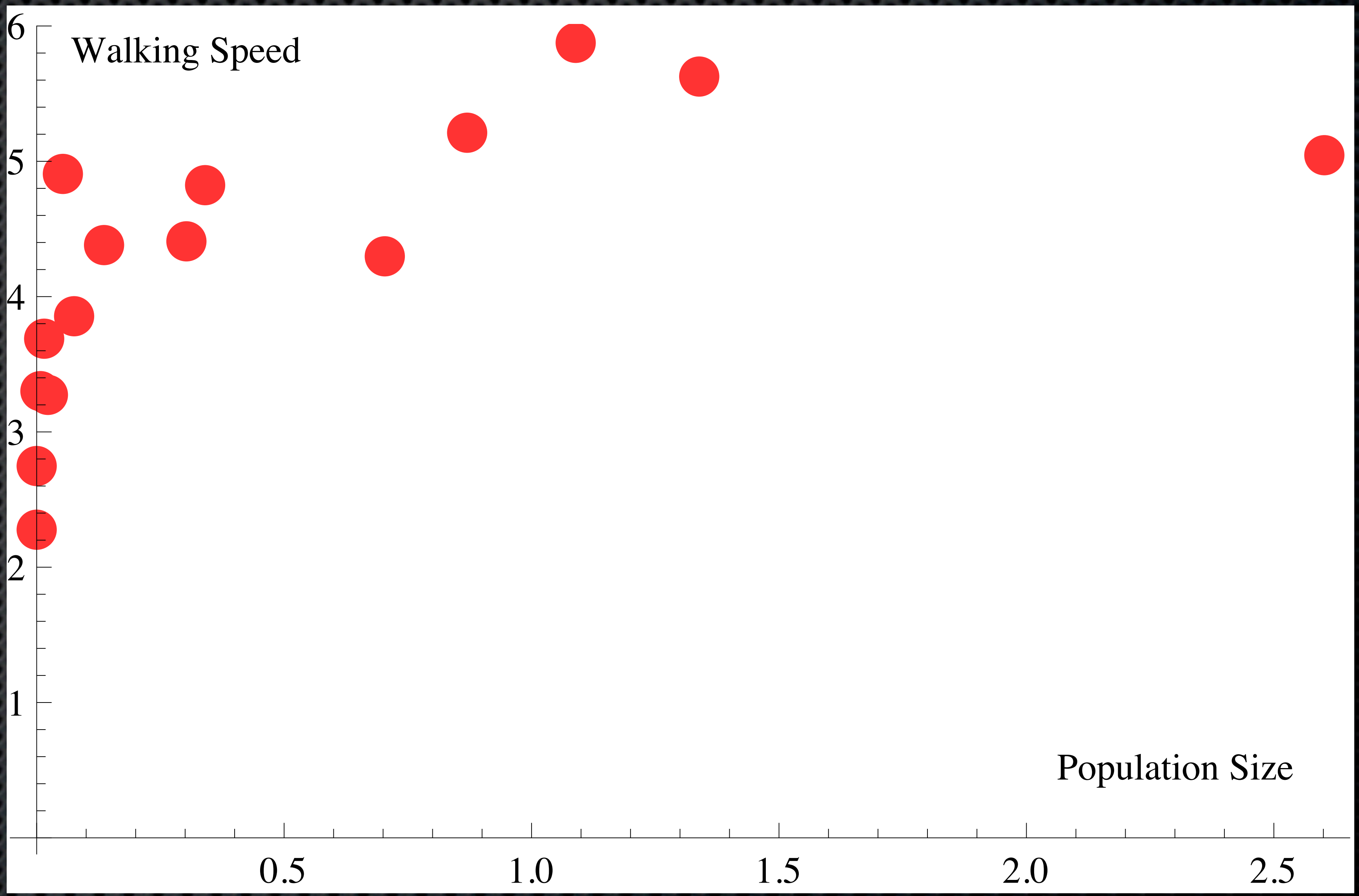


What Can We Assume?

- Many hidden influences
  - Economic and corporate structure prospects
  - Legal outcomes
- Some explicit influences
  - Time value of money
  - Detailed indenture
- We need something very simple

A Simple Credit Model

- Default τ in any time period ΔT is roughly proportional to its length
- But of course that must be conditional on P(t), surviving to time t
- $p(\tau \in [t,t+\Delta T] \mid \tau \geq t) \sim h\, \Delta T$ where h is called *hazard rate*
- Independence of time give us $p(\tau \in [t,t+\Delta T]) \,/\, p(\tau \geq t) \sim h\, \Delta T$

A Simple Credit Model

$$\frac{\Delta P(t)}{P(t)} = h\Delta t$$

$$\frac{dP(t)}{P(t)} = hdt$$

$$P(t) = e^{-ht}$$

Value A Cash Payment

- $V_0(0)$ = (Time Value of Money) × (Probability of Nothing Going Wrong)
- $V_1(0)$ = (Time Value of Money) × (Probability Something went Wrong)

  × (Value After Court Claims)

- $V(0) = V_0(0) + V_1(0)$

$$E[B(0,T)C|\tau > T]\,P(\tau > T) + E[B(0,\tau)C\delta|\tau \leq T]\,P(\tau \leq T)$$

Value A Cash Payment

$$E[B(0,T)C|\tau > T]\,P(\tau > T) + E[B(0,\tau)C\delta|\tau \le T]\,P(\tau \le T)$$

$$E\left[Ce^{\int_0^T r(s)ds}\,\middle|\,\tau > T\right]P(\tau > T) + E\left[C\delta e^{\int_0^\tau r(s)ds}\,\middle|\,\tau \le T\right]P(\tau \le T)$$

$$Ce^{-rT}P(\tau > T) + C\delta E\left[e^{-r\tau}\,\middle|\,\tau \le T\right]P(\tau \le T)$$

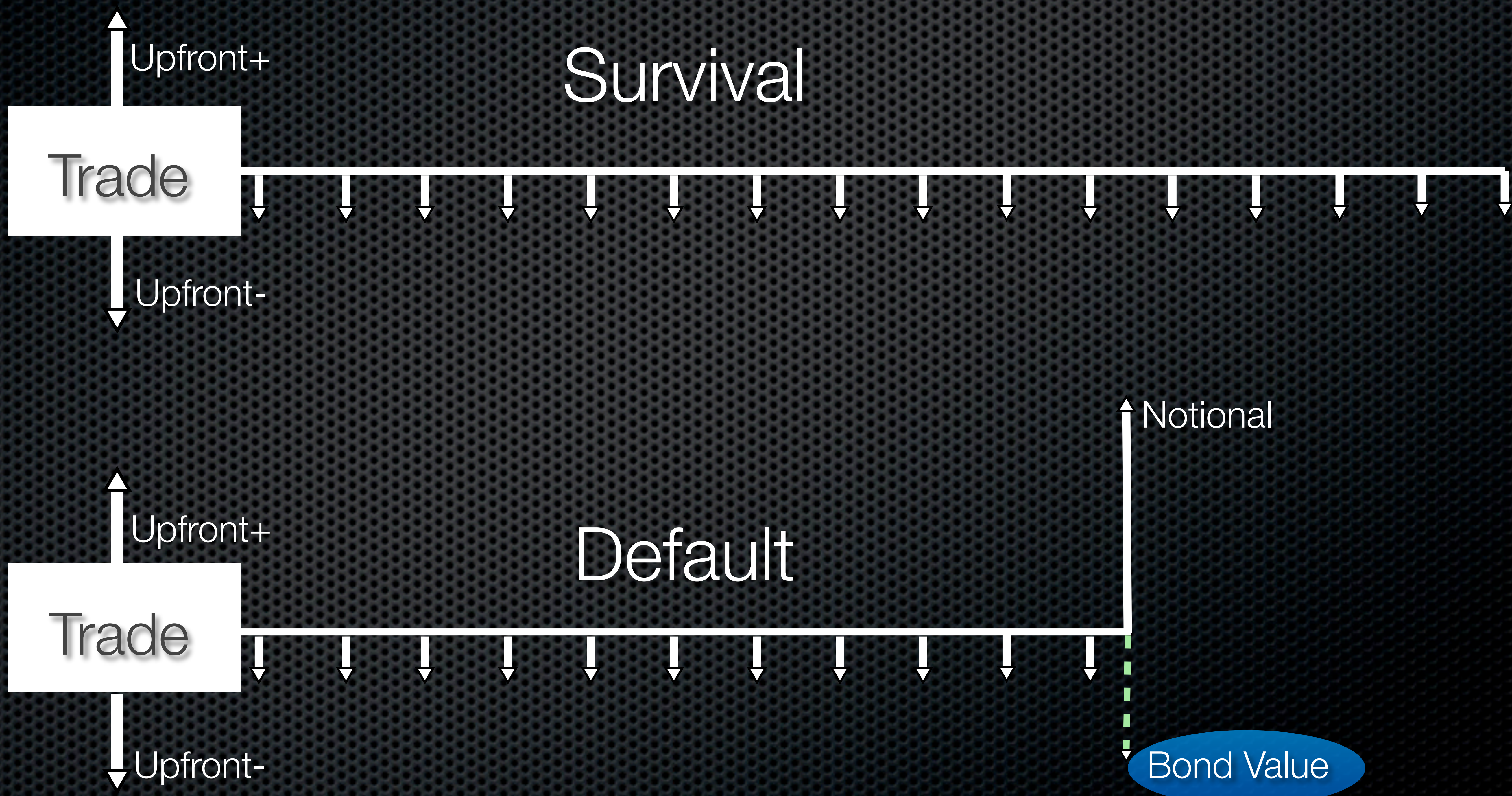
$$Ce^{-rT}e^{-hT} + C\delta\int_0^T e^{-r\tau}e^{-h\tau}h\,d\tau$$

$$Ce^{-(\underbrace{r+h}_{y})T} + C\delta\left(1 - e^{-(\underbrace{r+h}_{y})T}\right)\frac{h}{\underbrace{r+h}_{y}}$$

Credit Default Swaps (CDS)

- Designed as insurance on bonds
- Amenable to the same hazard valuation model as bond cashflows
- Regular premium payments, until maturity or default

CDS Cash Flows



Simpler Idea: Credit Options

- Pay x<\$1 right now
- Receive \$1 at time T in case of default before T
- No interest rate or recovery rate uncertainty

$$x = e^{-(r_T + h)T}$$

Credit Options

- Some attempts by exchanges to create them, but untraded
- Obvious relations to deep equity puts with strike K, equity recovery η

$$\text{Put} > \text{CreditOption} \times (K - \eta)$$

CDS Complications

- One equity, many CDS
  - Multiple entities, currencies
  - Formerly nonstandard coupons, tenors
- Discounting multiple cashflows
- Embedded cheapest-to-deliver options
- Daycount conventions

CDS Valuation (Approximate)

- Payment Leg

$$\int_0^T c\, e^{-(r+h)t} dt = \frac{c}{r+h}\left(1 - e^{-(r+h)T}\right)$$

- Receive Leg

$$\int_0^T (1-\delta) e^{-(r+h)t} h dt = \frac{h}{r+h}\left(1 - e^{-(r+h)T}\right)(1-\delta)$$

- Total

$$\frac{\left(1 - e^{-(r+h)T}\right)}{r+h}\left(h(1-\delta) - c\right)$$

CDS Valuation (Approximate)

- "Fair" when payment and receive leg have equal expected value

$$\frac{\left(1 - e^{-(r+h)T}\right)}{r+h}\left(h(1-\delta) - c\right) = 0 \implies c = h(1-\delta)$$

- We call this fair coupon the CDS spread *s*. Take L=1-δ, then

$$h = \frac{s}{L}$$

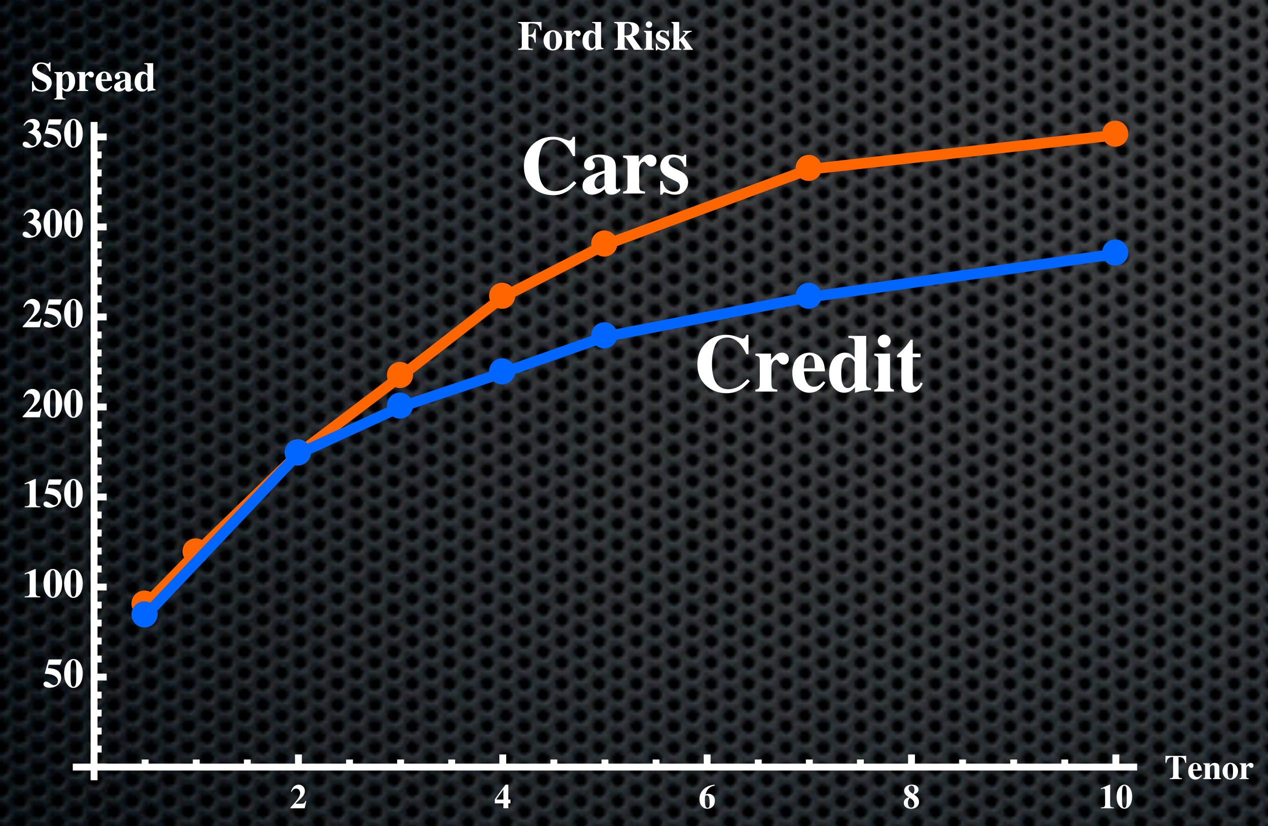
CDS Valuation (Approximate)

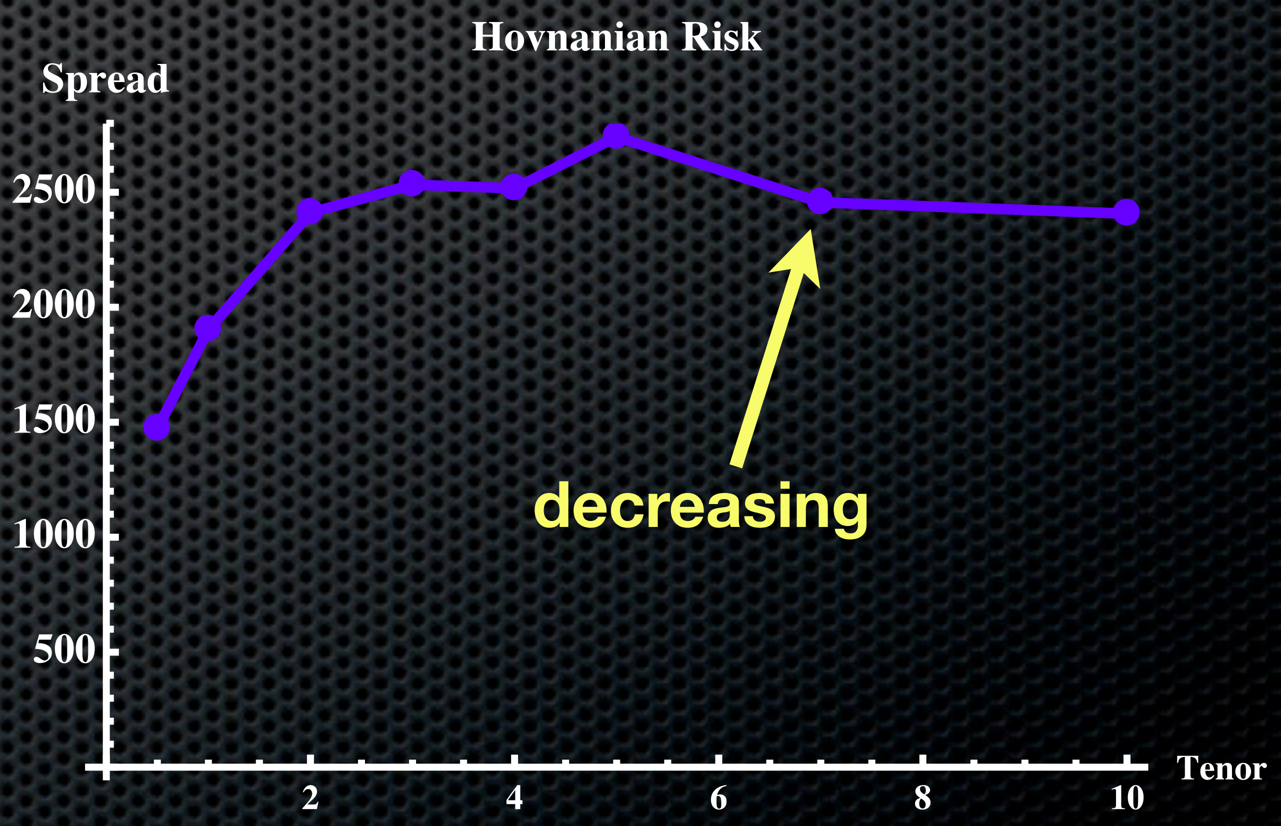
$$\frac{\left(1 - e^{-\left(r + \frac{s}{L}\right)T}\right)}{r + \frac{s}{L}} (s - c)$$

- Low price can mean
  - High recovery rate
  - Low default probability

Credit Curves: Analogy To Interest Rates

- Hazard rates can be taken to follow a curve, just like a yield curve
- Available instruments for calibration are sparser



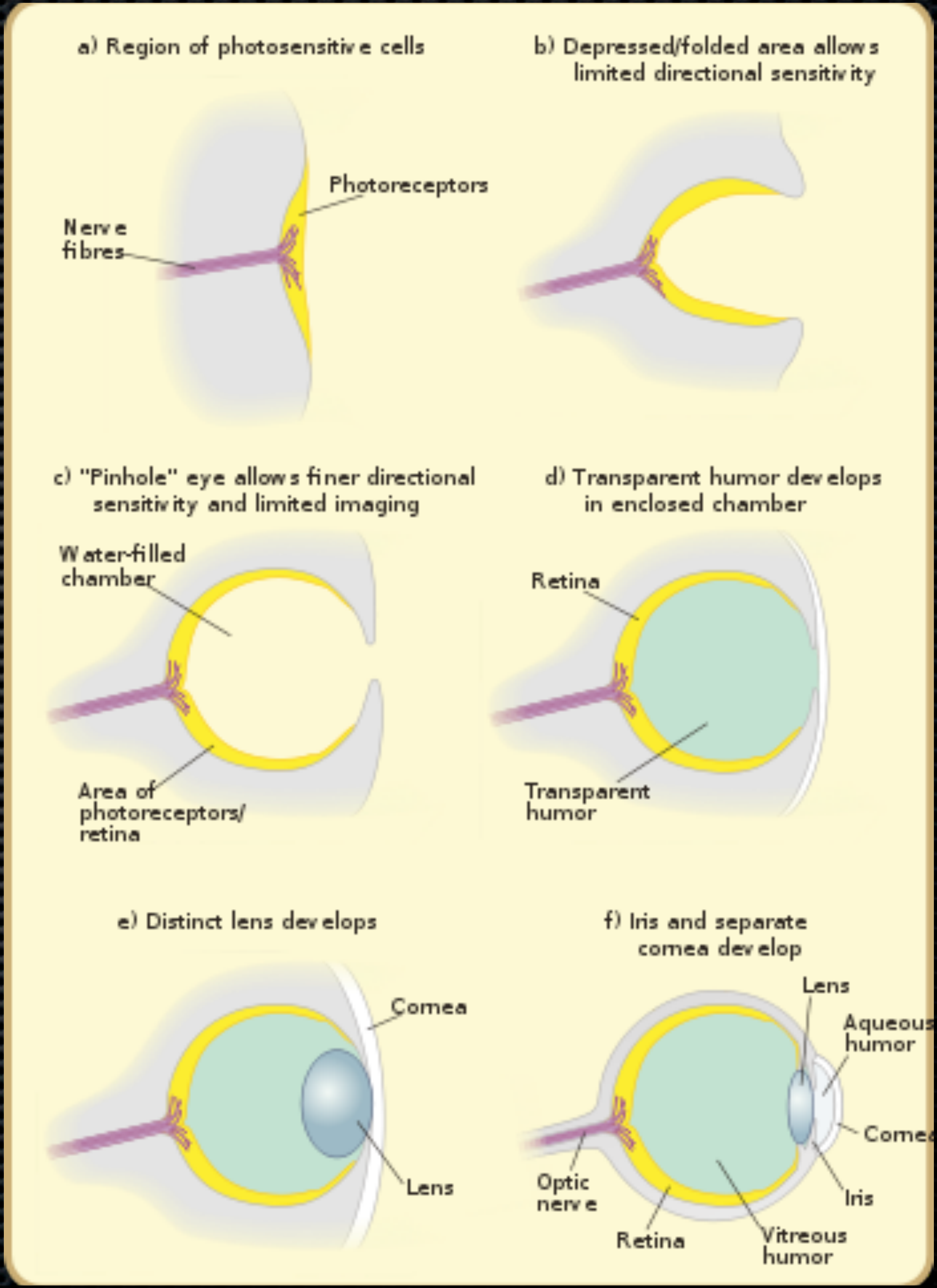


CDS Valuation (Precise)

- Hellish list of special cases and market conventions
  - Daycount conventions
  - Accrual
  - Settlement conventions
- A good flavor can be obtained from http://www.cdsmodel.com

Progress is Not Perfection

Improving a model can be valuable even if it still works poorly



Basic Assets And Valuation

- Equity pricing
  - Considered in the derivatives world to be fair, so why does anyone trade equities?
  - Consequences of dividends
- Similar situation for bonds, especially coupon bonds, and even exotics such as VIX futures
- If underlying asset value is unknown, what hope for derivatives?

Bonds and Loans

- Sum of cash payments (*coupons*) and final notional payment
- Fixed-rate bonds
  - Most common
  - Each coupon known at issue
- Floating-rate bonds
  - Each coupon tied to a benchmark such as Libor
  - Corporate loans suffer prepayment risk

The Tools of Credit Trading

Fixed-Coupon Bond Value

- Separable sum of individual payment values

$$\sum_{i=1}^{N} C_i \left(e^{-(r+h)T_i} + \delta \left(1 - e^{-(r+h)T_i} \right) \frac{h}{r+h} \right)$$

- Particularly simple if δ=0

$$\sum_{i=1}^{N} C_i e^{-(r+h)T_i}$$

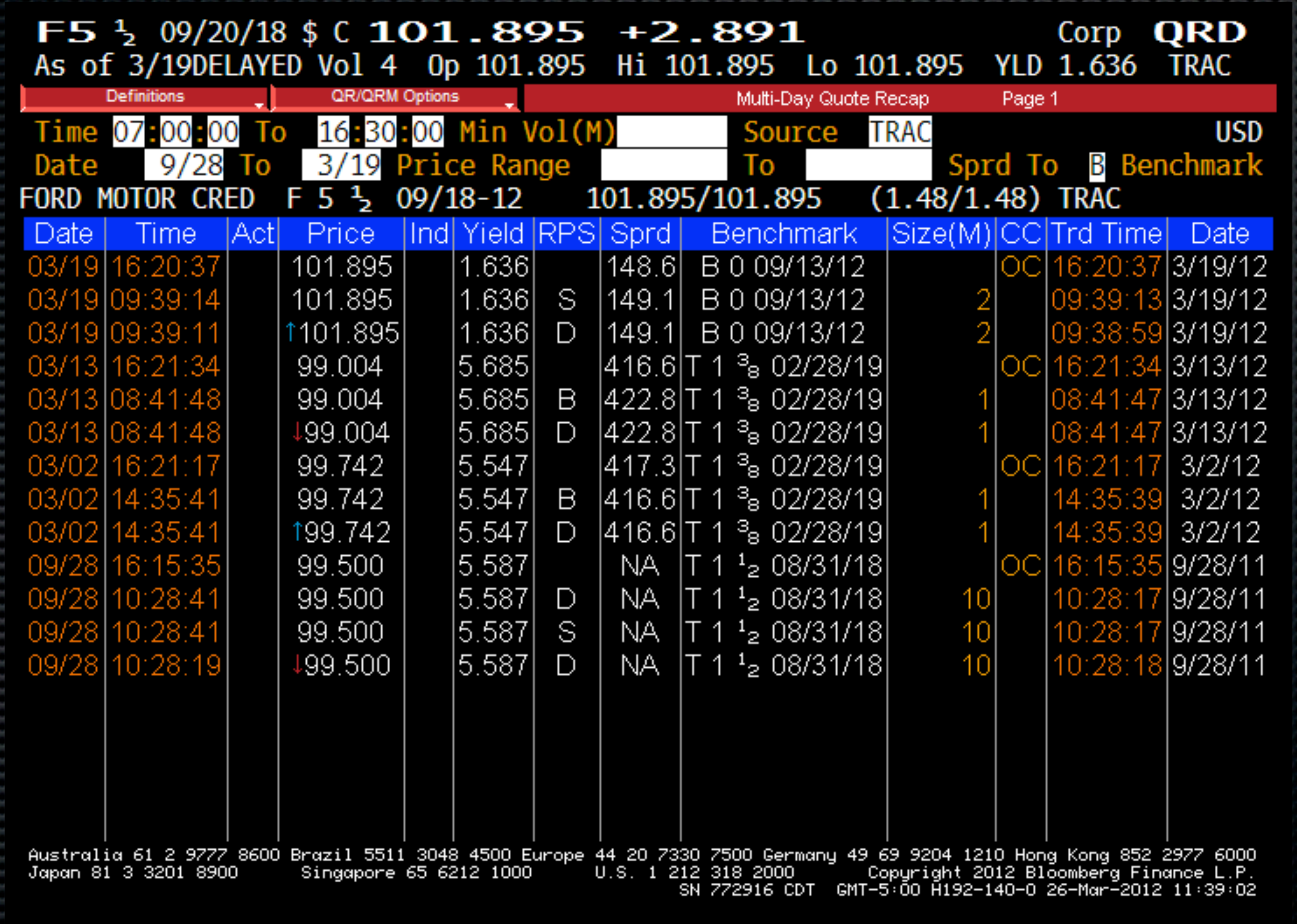
Cheap Tricks

- These bonds have prices *P*
- Implied spread

$$s(P) := s \ni \left\{ P = \sum_{i=1}^{N} C_i e^{-(s+r_i)T_i} \right\}$$

(must use a root finder)

Credit Markets Have Poor Liquidity



F5 $\frac{1}{2}$ 09/20/18 $ C **101.895 +2.891** Corp **QRD**
As of 3/19DELAYED Vol 4 Op 101.895 Hi 101.895 Lo 101.895 YLD 1.636 TRAC

Definitions | QR/QRM Options | Multi-Day Quote Recap Page 1

Time 07:00:00 To 16:30:00 Min Vol(M) Source TRAC USD
Date 9/28 To 3/19 Price Range To Sprd To B Benchmark
FORD MOTOR CRED F 5 $\frac{1}{2}$ 09/18-12 101.895/101.895 (1.48/1.48) TRAC

Date	Time	Act	Price	Ind	Yield	RPS	Sprd	Benchmark	Size(M)	CC	Trd Time	Date
03/19	16:20:37		101.895		1.636		148.6	B 0 09/13/12		OC	16:20:37	3/19/12
03/19	09:39:14		101.895		1.636	S	149.1	B 0 09/13/12	2		09:39:13	3/19/12
03/19	09:39:11		↑101.895		1.636	D	149.1	B 0 09/13/12	2		09:38:59	3/19/12
03/13	16:21:34		99.004		5.685		416.6	T 1 $\frac{3}{8}$ 02/28/19		OC	16:21:34	3/13/12
03/13	08:41:48		99.004		5.685	B	422.8	T 1 $\frac{3}{8}$ 02/28/19	1		08:41:47	3/13/12
03/13	08:41:48		↓99.004		5.685	D	422.8	T 1 $\frac{3}{8}$ 02/28/19	1		08:41:47	3/13/12
03/02	16:21:17		99.742		5.547		417.3	T 1 $\frac{3}{8}$ 02/28/19		OC	16:21:17	3/2/12
03/02	14:35:41		99.742		5.547	B	416.6	T 1 $\frac{3}{8}$ 02/28/19	1		14:35:39	3/2/12
03/02	14:35:41		↑99.742		5.547	D	416.6	T 1 $\frac{3}{8}$ 02/28/19	1		14:35:39	3/2/12
09/28	16:15:35		99.500		5.587		NA	T 1 $\frac{1}{2}$ 08/31/18		OC	16:15:35	9/28/11
09/28	10:28:41		99.500		5.587	D	NA	T 1 $\frac{1}{2}$ 08/31/18	10		10:28:17	9/28/11
09/28	10:28:41		99.500		5.587	S	NA	T 1 $\frac{1}{2}$ 08/31/18	10		10:28:17	9/28/11
09/28	10:28:19		↓99.500		5.587	D	NA	T 1 $\frac{1}{2}$ 08/31/18	10		10:28:18	9/28/11

Credit Markets Have Poor Liquidity

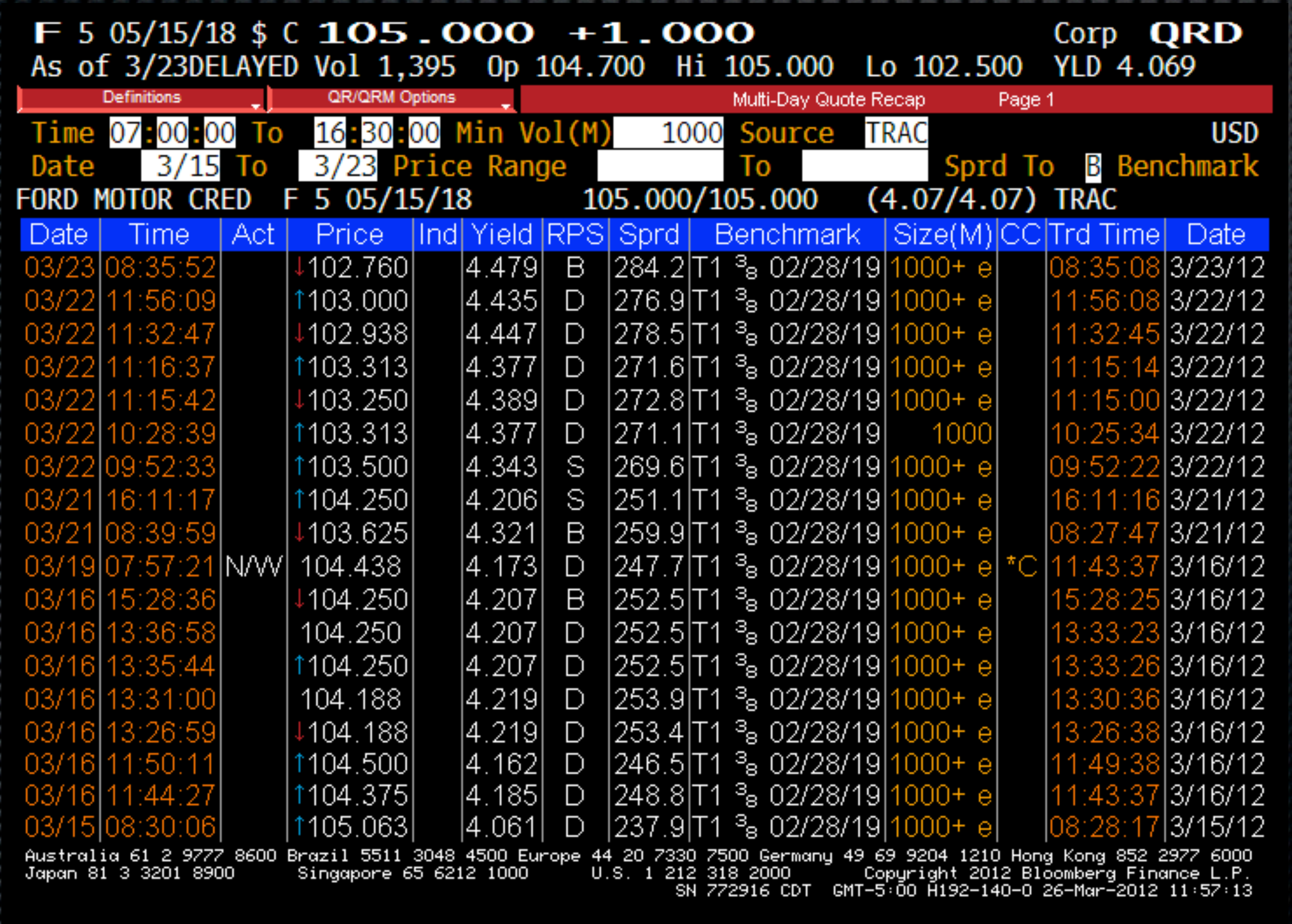
F5 ½ 09/20/18 $ C 102.535 +.535 Corp **QRD**
As of 2/21DELAYED Vol 20 Op 102.535 Hi 102.535 Lo 102.435 YLD 5.041 TRAC

Definitions | QR/QRM Options | Multi-Day Quote Recap Page 1

Time 07:00:00 To 16:30:00 Min Vol(M) Source TRAC USD
Date 12/13 To 2/21 Price Range To Sprd To B Benchmark
FORD MOTOR CRED F 5 ½ 09/18-12 102.535/102.535 (0.16/0.16) TRAC

Date	Time	Act	Price	Ind	Yield	RPS	Sprd	Benchmark	Size(M)	CC	Trd Time	Date
02/21	16:21:27		↑102.535		5.041		491.7	B 0 08/16/12		OC	16:21:27	2/21/12
02/21	13:41:47		↓102.435		5.059	D	493.4	B 0 08/16/12	10		13:41:41	2/21/12
02/21	13:41:41		↑102.535		5.041	S	491.7	B 0 08/16/12	10		13:41:46	2/21/12
02/17	16:21:17		102.000		5.137		502.0	B 0 08/16/12		OC	16:21:17	2/17/12
02/17	15:06:19		102.000		5.137	S	502.0	B 0 08/16/12	25		14:58:34	2/17/12
02/17	15:05:59		↑102.000		5.137	S	502.0	B 0 08/16/12	25		14:58:34	2/17/12
02/17	12:57:51		↓100.750		5.362	B	524.5	B 0 08/16/12	50		12:57:49	2/17/12
02/17	12:57:50		↓101.150		5.290	D	517.3	B 0 08/16/12	50		12:57:49	2/17/12
02/15	16:21:25		↑102.240		5.094		497.0	B 0 08/16/12		OC	16:21:25	2/15/12
02/15	11:46:21		↓102.140		5.112	D	498.2	B 0 08/16/12	10		11:46:11	2/15/12
02/15	11:46:12		↑102.240		5.094	S	496.4	B 0 08/16/12	10		11:46:17	2/15/12
02/09	16:20:16		99.200		5.646		422.2	T 1 ¼ 01/31/19		OC	16:20:16	2/9/12
02/09	11:58:32		↓99.200		5.646	B	420.1	T 1 ¼ 01/31/19	20		11:55:00	2/9/12
02/09	11:58:32		↓100.200		5.462	D	535.3	B 0 08/09/12	20		11:55:00	2/9/12
02/06	16:04:53	X	99.650		N.A.	D	NA	T 1 ½ 08/31/18	75	*X	09:17:39	9/16/11
02/06	07:02:07	A/O	99.650		N.A.	D	NA	T 1 ½ 08/31/18	75	SD	09:17:39	9/16/11
12/13	16:15:40		101.250		3.808		NA	B 0 11/15/12		OC	16:15:40	12/13/11
12/13	13:13:37		↓101.250		3.808	D	NA	B 0 11/15/12	11		13:13:31	12/13/11

Credit Markets Have Poor Liquidity



F 5 05/15/18 $ C **105.000 +1.000** Corp **QRD**
As of 3/23DELAYED Vol 1,395 Op 104.700 Hi 105.000 Lo 102.500 YLD 4.069
Definitions | QR/QRM Options | Multi-Day Quote Recap Page 1
Time 07:00:00 To 16:30:00 Min Vol(M) 1000 Source TRAC USD
Date 3/15 To 3/23 Price Range To Sprd To B Benchmark
FORD MOTOR CRED F 5 05/15/18 105.000/105.000 (4.07/4.07) TRAC

Date	Time	Act	Price	Ind	Yield	RPS	Sprd	Benchmark	Size(M)	CC	Trd Time	Date
03/23	08:35:52		↓102.760		4.479	B	284.2	T1 3/8 02/28/19	1000+ e		08:35:08	3/23/12
03/22	11:56:09		↑103.000		4.435	D	276.9	T1 3/8 02/28/19	1000+ e		11:56:08	3/22/12
03/22	11:32:47		↓102.938		4.447	D	278.5	T1 3/8 02/28/19	1000+ e		11:32:45	3/22/12
03/22	11:16:37		↑103.313		4.377	D	271.6	T1 3/8 02/28/19	1000+ e		11:15:14	3/22/12
03/22	11:15:42		↓103.250		4.389	D	272.8	T1 3/8 02/28/19	1000+ e		11:15:00	3/22/12
03/22	10:28:39		↑103.313		4.377	D	271.1	T1 3/8 02/28/19	1000		10:25:34	3/22/12
03/22	09:52:33		↑103.500		4.343	S	269.6	T1 3/8 02/28/19	1000+ e		09:52:22	3/22/12
03/21	16:11:17		↑104.250		4.206	S	251.1	T1 3/8 02/28/19	1000+ e		16:11:16	3/21/12
03/21	08:39:59		↓103.625		4.321	B	259.9	T1 3/8 02/28/19	1000+ e		08:27:47	3/21/12
03/19	07:57:21	N/W	104.438		4.173	D	247.7	T1 3/8 02/28/19	1000+ e	*C	11:43:37	3/16/12
03/16	15:28:36		↓104.250		4.207	B	252.5	T1 3/8 02/28/19	1000+ e		15:28:25	3/16/12
03/16	13:36:58		104.250		4.207	D	252.5	T1 3/8 02/28/19	1000+ e		13:33:23	3/16/12
03/16	13:35:44		↑104.250		4.207	D	252.5	T1 3/8 02/28/19	1000+ e		13:33:26	3/16/12
03/16	13:31:00		104.188		4.219	D	253.9	T1 3/8 02/28/19	1000+ e		13:30:36	3/16/12
03/16	13:26:59		↓104.188		4.219	D	253.4	T1 3/8 02/28/19	1000+ e		13:26:38	3/16/12
03/16	11:50:11		↑104.500		4.162	D	246.5	T1 3/8 02/28/19	1000+ e		11:49:38	3/16/12
03/16	11:44:27		↑104.375		4.185	D	248.8	T1 3/8 02/28/19	1000+ e		11:43:37	3/16/12
03/15	08:30:06		↑105.063		4.061	D	237.9	T1 3/8 02/28/19	1000+ e		08:28:17	3/15/12

Australia 61 2 9777 8600 Brazil 5511 3048 4500 Europe 44 20 7330 7500 Germany 49 69 9204 1210 Hong Kong 852 2977 6000
Japan 81 3 3201 8900 Singapore 65 6212 1000 U.S. 1 212 318 2000
SN 772916 CDT GMT-5:00 H192-140-0 26-Mar-2012 11:57:13

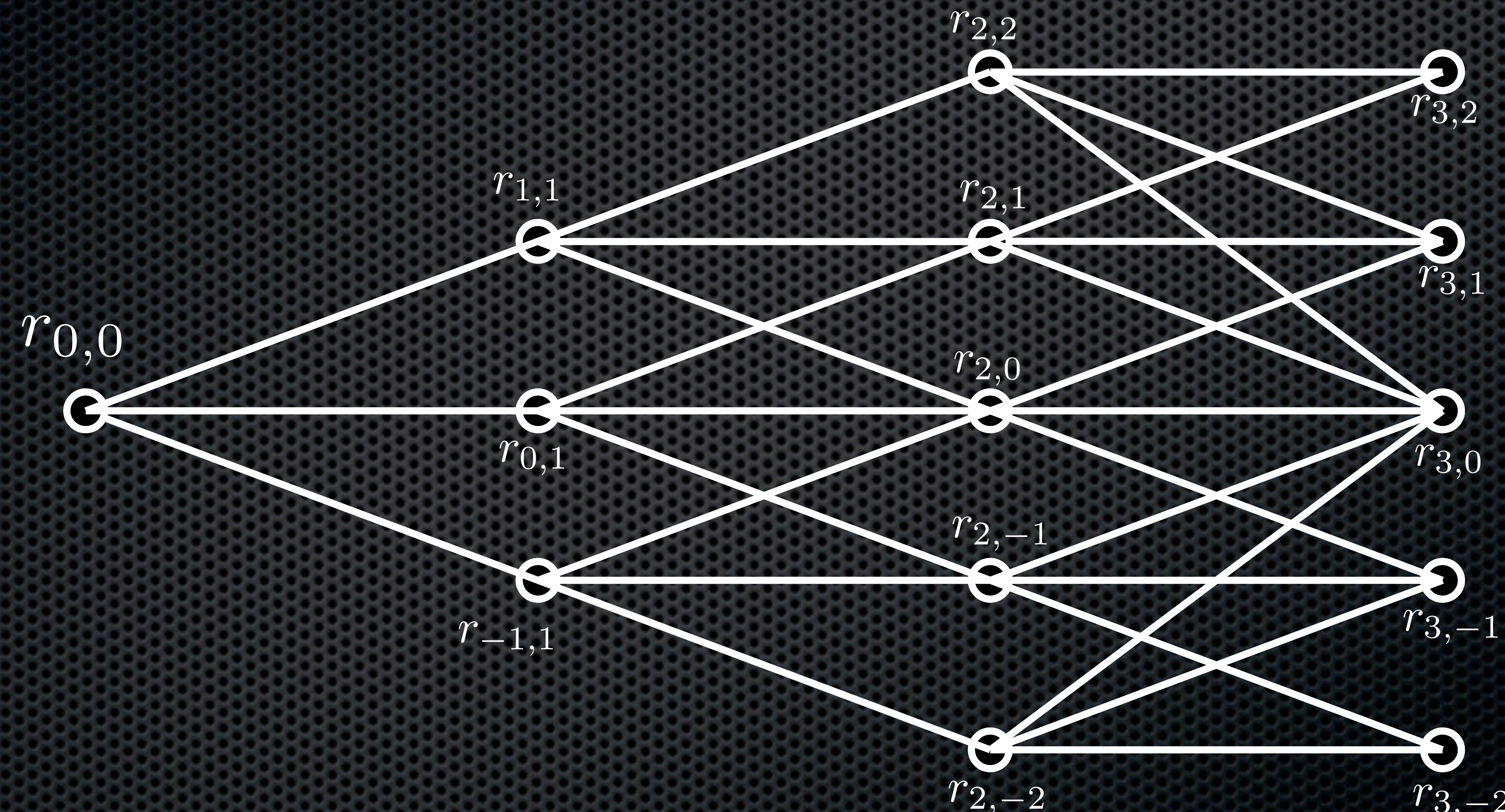
Modeling Consequences of Illiquidity

- Model fits use modified objective funtions
- Bid/offer widths characterize practical useful limit of model accuracy
- Models that can incorporate more liquid indicators are highly desirable
    - Index betas
    - Factor models

Optionality in Bonds

- Most issues have embedded prepayment options
- Typically viewed as options on interest rate
- Equally important is the option on creditworthiness

Pricing Embedded Optionality



Embedded Bond Options

- Calls
  - Issuer has option to buy the bond back
  - Example: Issuer may pay 103% of notional to retire the bond in 2015
- Puts
  - Holder has option to force issuer to buy the bond back
  - Example: Holder may force the issuer to buy the bond for 100% of notional in 2015

Common Embedded Option Models

Black-Derman-Toy	106 bp
Linear Gauss Markov	108 bp
Lognormal Swaption	121 bp
Generalized Vasicek	133 bp

Terminology in Bond Trading

- Option Adjusted Spread $s \ni V_{\mathrm{Mkt}} = V(r(t) + s, \sigma_r, \mu_r)$
- Duration $-\dfrac{dV/dy}{V}$
- Yield
  - Yield to Call
  - Yield to Worst
- DV01 $-\dfrac{dV}{ds} \times 10^4$

Other Instruments

- Credit Indexes
- Bond ETFs
- Collateralized Securities

Counterparty Risk

Valuing A Risky Put

- Put on *A* purchased from *B* with default time τ_B

Probability density

Survival past option tenor

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(\tau_B, S_A)\mathbb{1}_{\{\tau_B > T\}}(K - S_A)^+ \, d\tau_B \, dS_A$$

(potentially separable integrals)

** forward value*

Valuing A Risky Put, Using A Copula

Bivariate Normal Distribution Relates *A* and *B*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(w, z; \rho)\mathbb{1}_{\{\tau_B(w)>T\}}\,(K - S_A(z))^{+}\,dz\,dw$$

Pearson correlation completely characterizes dependence of z, w

Valuing A Risky Put, Using A Copula

Restrict to non-default domain

$$\int_{N^{-1}(1-e^{-hT})}^{\infty} \int_{-\infty}^{\infty} p(w, z; \rho) \left(K - S_A(z)\right)^{+} dz\, dw$$

$$\text{with} \quad S_A(z) = S_0 \exp\left((r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T} z \right)$$

Valuing A Risky Put, Using A Copula

Further restrict to payoff domain

$$\int_{N^{-1}(1-e^{-hT})}^{\infty} \int_{-\infty}^{\frac{\log K/S_A(z)-(r-\sigma^2/2)T}{\sigma\sqrt{T}}} p(w,z;\rho)\,(K - S_A(z))\,dz\,dw$$

Bivariate Normal Density Is Separable

$$p(w, z; \rho) = n(w; 0, 1)\ p_{z|w}(w, z)$$
$$= n(w; 0, 1)\ n(z; \rho w, \sqrt{1 - \rho^2})$$

Valuing A Risky Put, Using A Copula

Separated form

$$\int_{N^{-1}(1-e^{-hT})}^{\infty} n(w) \int_{-\infty}^{\frac{\log \frac{K}{S_A(z)} - (r-\sigma^2/2)T}{\sigma\sqrt{T}}} n(z; \rho w, \sqrt{1-\rho^2})\,(K - S_A(z))\, dz dw$$

Valuing A Risky Put, Using A Copula

Black Scholes-like interior integral

$$\int_{N^{-1}(1-e^{-hT})}^{\infty} n(w) \int_{-\infty}^{\frac{\log \frac{K}{S_A\left(z\sqrt{1-\rho^2}+\rho w\right)} - (r-\sigma^2/2)T}{\sigma\sqrt{T}}} n(z)\left(K - S_A(z\sqrt{1-\rho^2}+\rho w)\right) dz\,dw$$

$$= \int_{N^{-1}(1-e^{-hT})}^{\infty} n(w) BS\left(S_0, K, \sigma\sqrt{1-\rho^2}, r, \frac{w\rho\sigma}{\sqrt{T}} - \frac{1}{2}\sigma^2\rho^2\right) dw$$

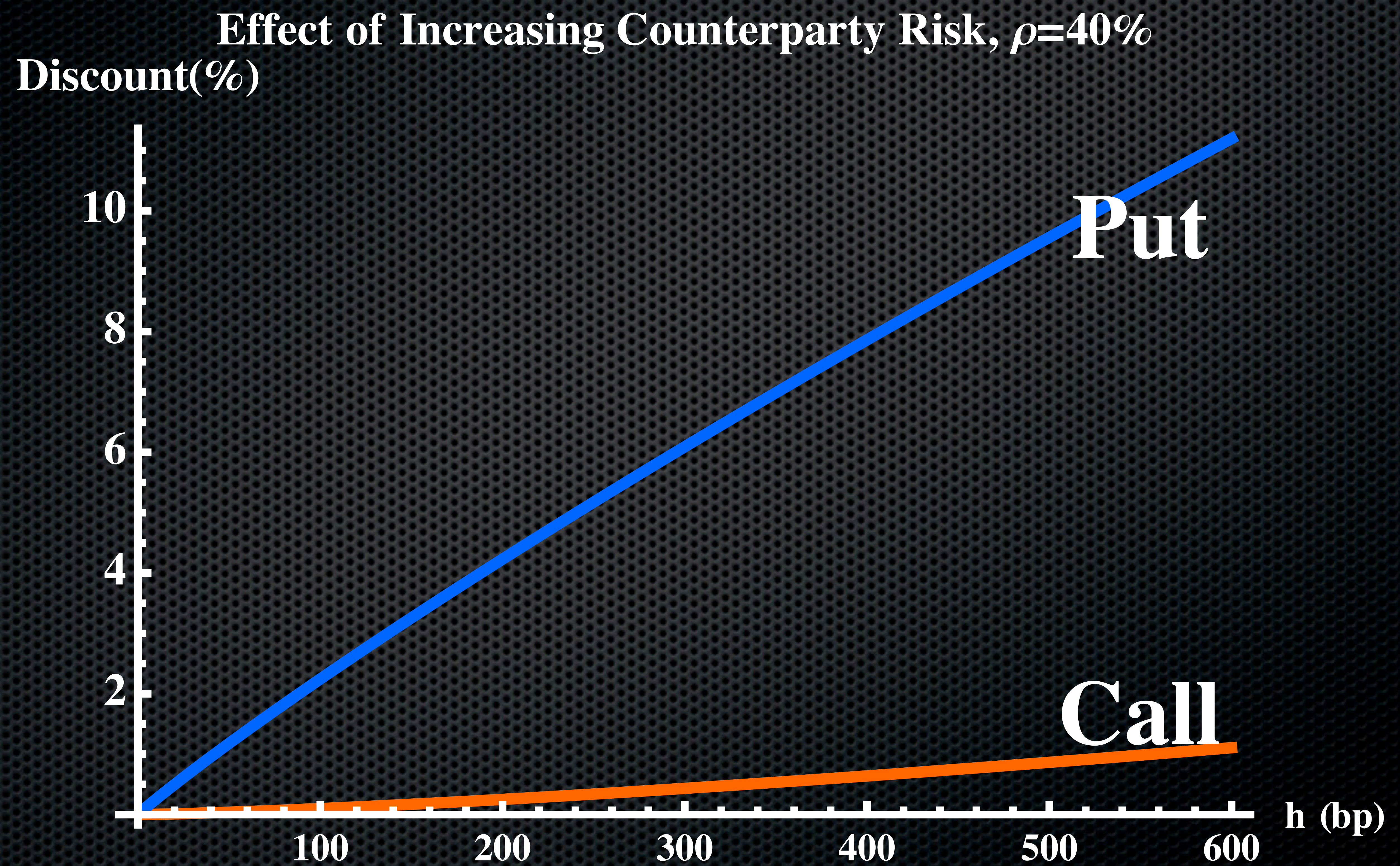
volatility adjustment

drift adjustment

Amenable To Deterministic Quadrature

- Integration against normal and exponential kernels is well understood in one dimension
- Numerical efficiency is very high
- Especially important if we are accomodating skew in the terminal probability distribution

Vanilla Option Discount



Sources of Credit Data

- Trading Broker Relationships
  - Phone calls
  - Mail-shots, Bloomberg messages
- Aggregators
  - Live: Bloomberg, Markit, CMA
  - Historical: FactSet, Moodys
- Data specialists

Structural Models

- Merton
  - Simple option-like view
  - Limited perspective on debt horizon
- CreditGrades
  - May already have defaulted (!?)
  - Inconsistency in calibration
- Both models rely on effectively hidden parameters

Merton Model

- Economic value of assets *A*
  - Differs from accounting/book value
  - Follows a geometric brownian motion
- Fixed effective “strike” *L*
  - Economic value of liabilities
  - Debt, accounts payable
- Equity value *S* is option on *A* with strike *L*

Merton Model

$$S = BS(A_0, L, \sigma_A, r, q_A, T)$$

Unobservable

Specific tenor

Merton Model (Calibration)

- Basic principle of *all* calibration: observations must outnumber variables
  - Technical requirement: time series of $N \geq 4$ equity prices
  - Practical requirement: option prices or many asset prices
- Calibration
  - Historical asset prices
  - Option prices

Merton Model (Parametric Calibration)

- Instantaneous asset to equity correlation implied by Ito Lemma

$$dS = (\text{drift terms})\, dt + \frac{\partial S}{\partial A} A \sigma_A dz$$

$$\sigma = \frac{A}{E} N(d_1) \sigma_A$$

Merton Model (Parametric Calibration)

- Fixed point attractor eliminates need for two dimensional solver

$$\sigma_A^{(0)} := \sigma$$

$$A^{(n+1)} := A \ni S = BS\Big(A, L, \sigma_A^{(n)}, r, q_A, T\Big)$$

$$\sigma_A^{(n+1)} := \sigma \frac{S}{A^{(n+1)}} \frac{1}{N\left(d_1(\sigma_A^{(n)}, A^{(n+1)})\right)}$$

Merton Model Variations

- Jumps, stochastic volatility
- Stochastic barrier
- Multilevel outcomes (credit ratings)
- Distance to default (risk cohorts)

$$\mathrm{DD} = \frac{\log A - \log L}{\sigma_A}$$

CreditGrades

- Stochastic process for assets similar in spirit to Merton
- Random default threshold, versus average recovery rate

$$\bar{L}e^{\lambda Z - \frac{1}{2}\lambda^2}$$

- Default probability (approximate) $c_t^2 := \sigma_A^2 + \lambda^2, \quad b_t := \frac{1}{\bar{L}} V_0 e^{\lambda^2}$

$$N\left(-\frac{c_t}{2} + \frac{\log b_t}{c_t}\right) - b_t N\left(-\frac{c_t}{2} - \frac{\log b_t}{c_t}\right)$$

CG Dynamics: Shifted Lognormal

$$\frac{dA}{A} = (r - q_A)dt + \sigma dz$$

$$dS = (r - q_A)Sdt + \sigma(S + L)dz$$

$$L_T = L_0 e^{rT}$$

$$p(S_T) = \frac{1}{\sigma\sqrt{2\pi T}(S_T + L_T)} \exp\left(-\frac{\log\frac{S_0+L_0}{S_T+L_T} + (r - q_A)T - \frac{1}{2}\sigma^2 T}{2\sigma\sqrt{T}}\right)$$

$$\text{Default if} \quad \frac{S_0 + L_0}{L_0} e^{-\frac{1}{2}\sigma^2\tau + z\sigma\sqrt{\tau}} < 1$$

CG Dynamics: Contingent Claims

$$rV = \frac{\partial V}{\partial t} + \frac{1}{2}\sigma(S+L)\frac{\partial^2 V}{\partial S^2} + (r - q_A)S\frac{\partial V}{\partial S}$$

$$C = C_{BS}(S+L, K+L) - \frac{S+L}{L}C_{BS}\left(\frac{L^2}{S+L}, K+L\right)$$

(down-and-out call, easily fitted skew)

CreditGrades: Issues

- Short-term defaults
  - “Solved” at time zero due to unknown default barrier
  - Unsolved at forward times (conditional on survival)
- Weren’t we supposed to be thinking of equity as an *option*?

Multidimensional Credit

- Portfolios
  - Counterparty Risk in Portfolios
  - Portfolios of Credit Instruments
- Collateralized Debt

Marginal Distributions

- One-dimensional single-variable distributions
  - Default time
  - Asset price
- Can be transformed to convenient forms using Radon-Nikodym derivatives
- Same as change-of-variables in integration

Distributional Transformation

- Ratio of continuous distributions is the Radon-Nikodym derivative
- Transformation to the unit uniform distribution uses cumulative distribution function (CDF)

$$P(s) = \int_{-\infty}^{s} p(x)dx \qquad [-\infty, \infty] \xrightarrow{P} [0, 1]$$

- Transforming from one distribution to another is a composition of a CDF and an inverse CDF

$$Q(s) = \int_{-\infty}^{s} q(x)dx \qquad [-\infty, \infty] \xrightarrow{P^{-1} \circ Q} [-\infty, \infty]$$

Important Densities

- Black-Scholes Density $$b(S) = \frac{n(d_2(S_0, S))}{Se^{rT}\sigma\sqrt{T}}$$
- Gaussian/normal density $$n(z)$$
- Hazard/Poisson Model Density $$he^{-hT}$$

Distributional Transformation Examples

- Black-Scholes to Gaussian

$$z = \frac{\log\left(S/S_0\right) - (r - \sigma^2/2)T}{\sigma\sqrt{T}} \qquad S = S_0 \exp((r - \sigma^2/2)T + \sigma\sqrt{T}z)$$

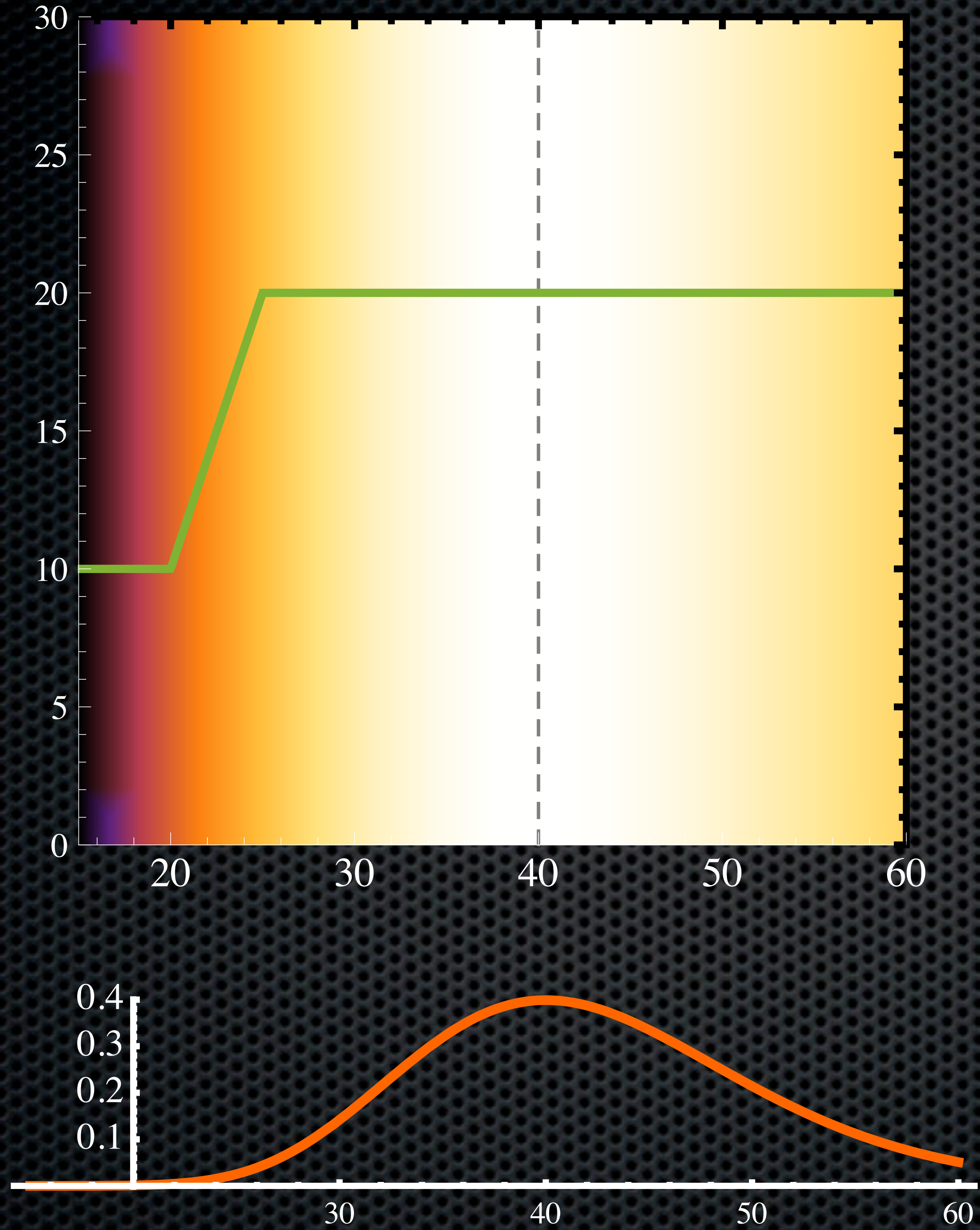
- Poisson to Gaussian

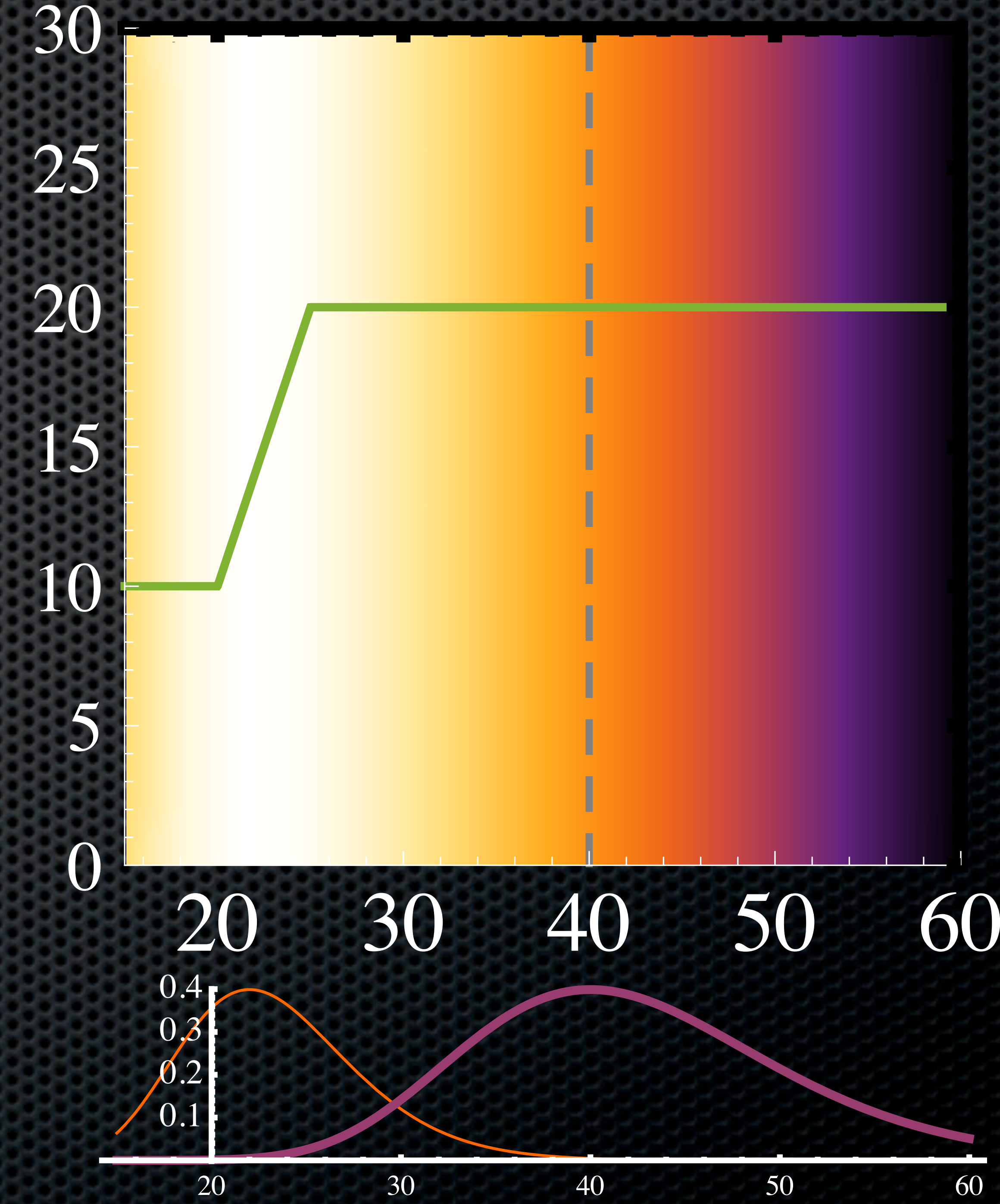
$$z = N^{-1}\left(1 - e^{-h\tau}\right) \qquad \tau = -\frac{\log(N(-z))}{h}$$

Importance Sampling

- Idea: Perform Monte Carlo sampling from distribution where "interesting" things happen
  - Added probability of "interesting" events means we adjust weights of samples afterwards to compensate
  - Obtain higher resolution in interesting regime to improve overall resolution
  - Weighting is same thing as Radon-Nikodym or change of variables
- Typically done after transforming everything to gaussian variables

Importance Sampling





Importance Sampling

- Weighting (or Radon-Nikodym Derivative) for single-variable gaussian

$$z' = z - a \implies w = e^{z - a + a^2/2}$$

- Can be taken as first dimension of a *copula*

Relating Marginals With Copulas

- We may have an idea about outcomes for any given individual asset, but how should we consider them related?
- Typical answer for equity or FX basket derivatives: the underlying Wiener processes are correlated

$$
\begin{aligned}
dA^{(1)} &= \mu_1 A^{(1)} dt + \sigma_1 A^{(1)} dW^{(1)} \\
dA^{(2)} &= \mu_2 A^{(2)} dt + \sigma_2 A^{(2)} dW^{(2)} \\
&\vdots \\
dA^{(d)} &= \mu_d A^{(d)} dt + \sigma_d A^{(d)} dW^{(d)}
\end{aligned}
$$

$$\left\langle dW^{(\ell)}, dW^{(k)} \right\rangle = \rho_{\ell,k}$$

Relating Marginals With Copulas

- The bias toward continuous processes led people to try correlating defaults with them for many years
- Consider instead the terminal distributions after macroscopic time T

$$z^{(\ell)} := \frac{1}{\sigma_\ell \sqrt{T}} \int_0^T dW^{(\ell)}, \quad \ell = 1, \ldots, d$$

- They share the correlation of the processes

$$\left\langle z^{(\ell)}, z^{(k)} \right\rangle = \rho_{\ell,k}$$

Relating Marginals With Copulas

- In many cases, including European exercise and counterparty risk, we are only concerned with events to a specific terminal time T

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(\tau_B, S_A)\mathbb{1}_{\{\tau_B > T\}}\,(K - S_A)^+\,d\tau_B dS_A$$

$$\int_{N^{-1}(1-e^{-hT})}^{\infty} n(w) BS\left(S_0, K, \sigma\sqrt{1-\rho^2}, r, \frac{w\rho\sigma}{\sqrt{T}} - \frac{1}{2}\sigma^2\rho^2 \right) dw$$

- In other cases, we may be willing to approximate by pretending we are concerned with one (or a small number) of discrete time(s)

Relating Marginals With Copulas

- By restricting to a single realization, we allow ourselves to relate marginal distributions (but not processes)
- This relationship is worked out by making distributional transformations to gaussian variables
- Relationships between gaussian variables are entirely characterized by the means and covariance matrix
- High dimensionality typically demands Monte Carlo techniques

Gaussian Monte Carlo Samples

- Univariate sample: if $z' \sim N(0,1)$ and $z = \sigma z' + \mu$ then $z \sim N(\mu, \sigma)$
- Assume we have a set of random variables z_i with correlation matrix **Σ**
- Constructing a random sample of them requires some linear algebra
- Since **Σ** is like σ^2, we need a matrix "square root" to do a similar trick. This is the *Cholesky Decomposition* **C**.

$$\vec{z} = \vec{\mu} + \vec{z'} \cdot C$$

Copulas For Loss Distributions

- A common case: a portfolio of N=100 credit instruments
- We have a payoff depending on total losses experienced before a time horizon T
- Need to relate the default times τ_n to each other
- Univariate relation to a gaussian variable z_n is simple

$$\tau_n = P^{-1}\left(N(z_n)\right) = -\frac{1}{h_n}\log\left(N(z_n)\right)$$

Copulas For Loss Distributions

- Obtaining a multivariate sample is also simple

$$\vec{\tau} = -\frac{1}{\vec{h}} \log\left(N(\vec{z})\right) \qquad \text{(operations taken elementwise)}$$

- Assume M samples of N-dimensional z' values. Then obtaining M samples of N-dimensional τ values is easy in vector languages

```
z = dot( w, chol(correls) )
tau = -log(1-cumnorm(z))/hazardRates
```

Copulas For Loss Distributions

- Each set of default times is an *N*-dimensional vector
- By comparing default times τ_n to horizon T in each of the *M* samples, we obtain *M* samples of the loss distribution L

```
z = dot( w, chol(correls) )
tau = -log(1-cumnorm(z))/hazardRates
losses = sum( exp(-r*tau)* indicator(tau<T))
```

Loss Distribution By Correlation

Application: Tranche Protection

- Actual loss level: L
- Payoffs
  - Equity: Min(L, A_{Eq})
  - Mezzanine: Min[Max(0, $L - A_{Eq}$), $A_{Mezz} - A_{Eq}$]
  - Senior: Max(0, $L - A_{Mezz}$)

Buying all three completely covers all losses L

Application: Tranche Protection

- Typical to assume “constant correlation”
- Also often assume constant hazard rates
  - Portfolio members tend to be similar
  - Historical lookup tables similar for all members

Copulas and Importance Sampling

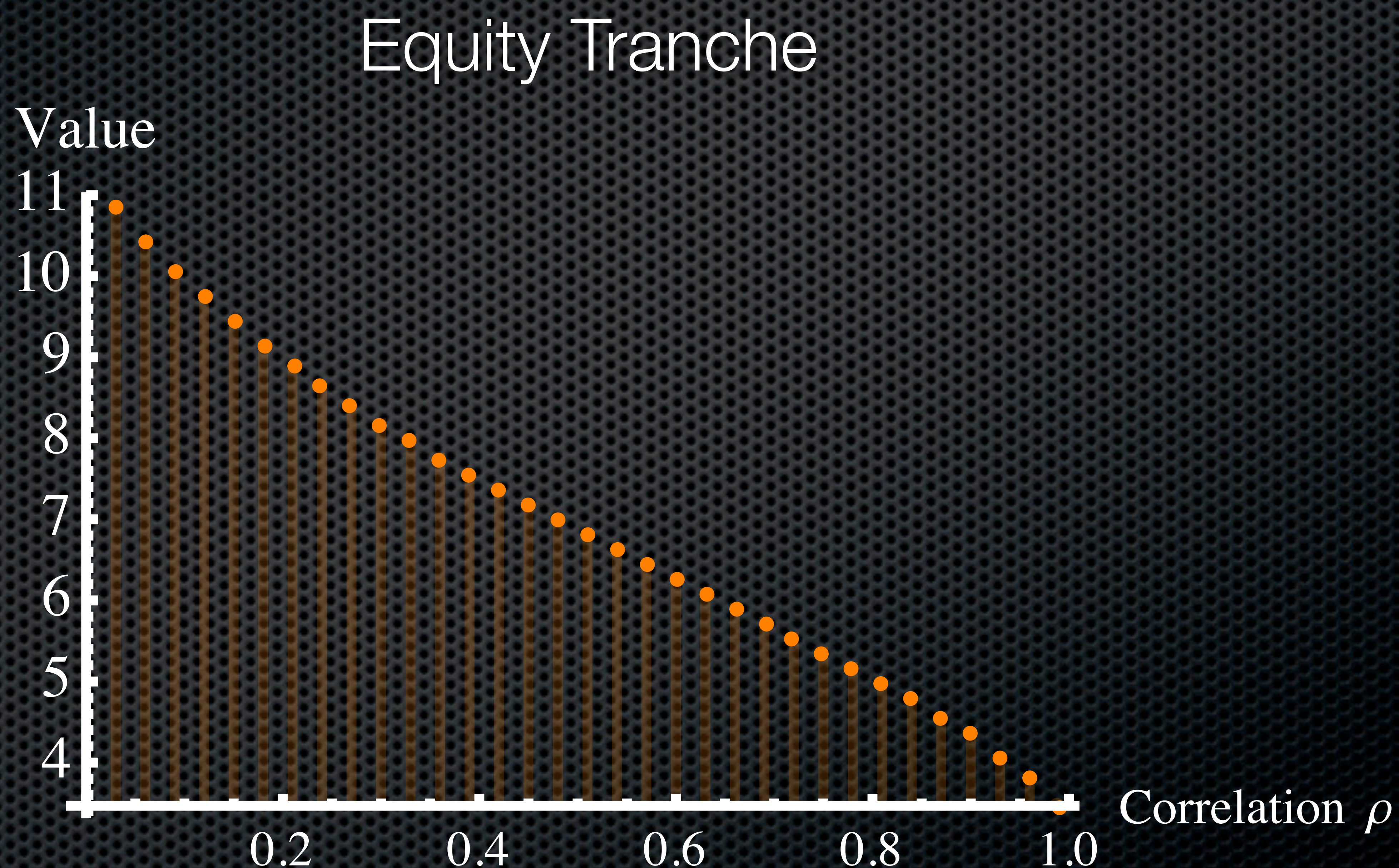
- Defaults are rare events
- Importance sampling can greatly increase accuracy at tiny cost, by generating extra samples with nonzero losses
- The use of a gaussian copula makes it very easy
  - Especially if importance sampling only one dimension
  - Radon-Nikodym derivative gets more complex otherwise

Copula Problems

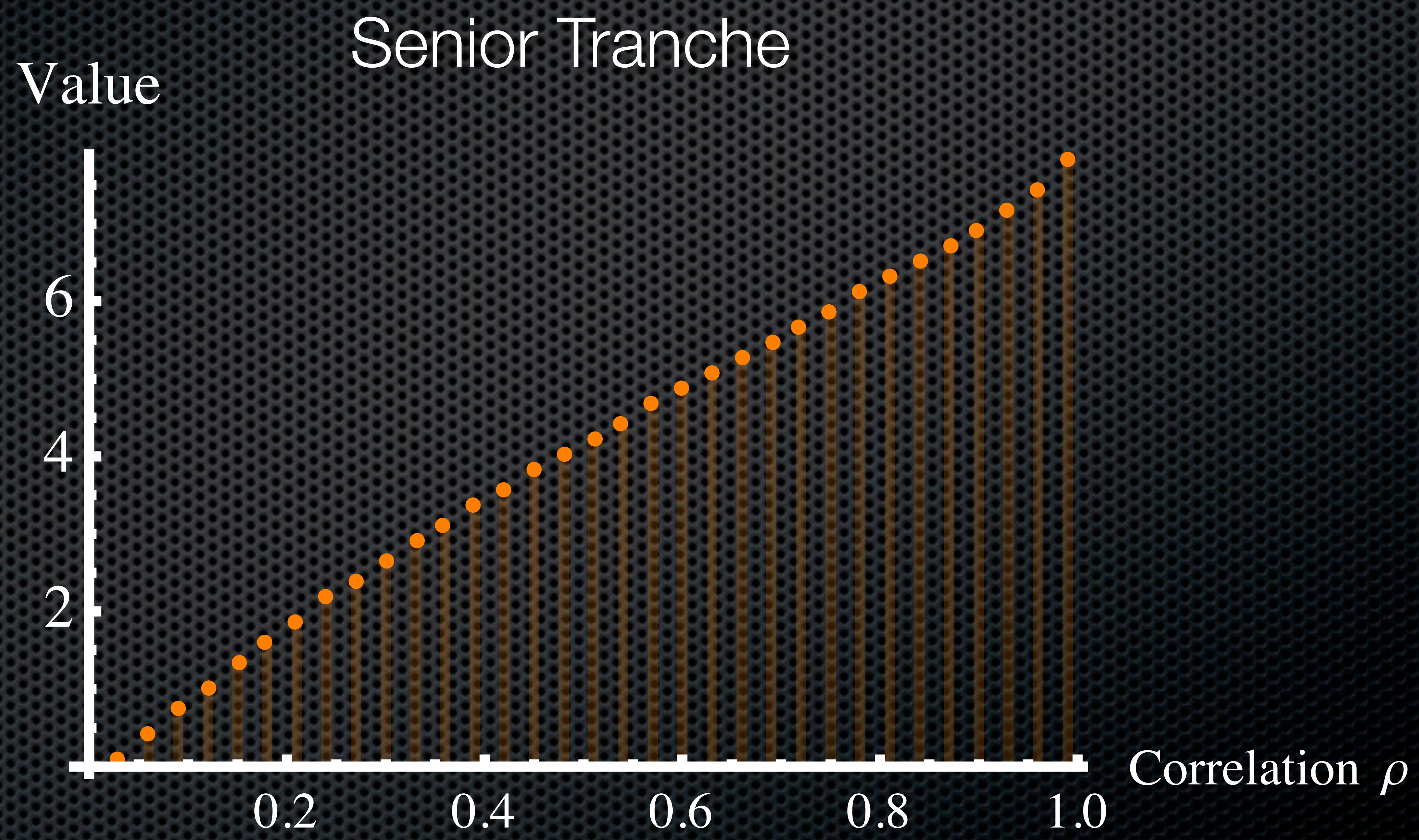
- Where should correlations be obtained?
  - Are equity correlations relevant?
  - Can imply a constant correlation similar to volatility in BS
- New problem: Mezzanine tranche protection price is not monotonic in “constant” correlation

$$\rho_{\ell,k} = \rho \qquad \ell \neq k$$

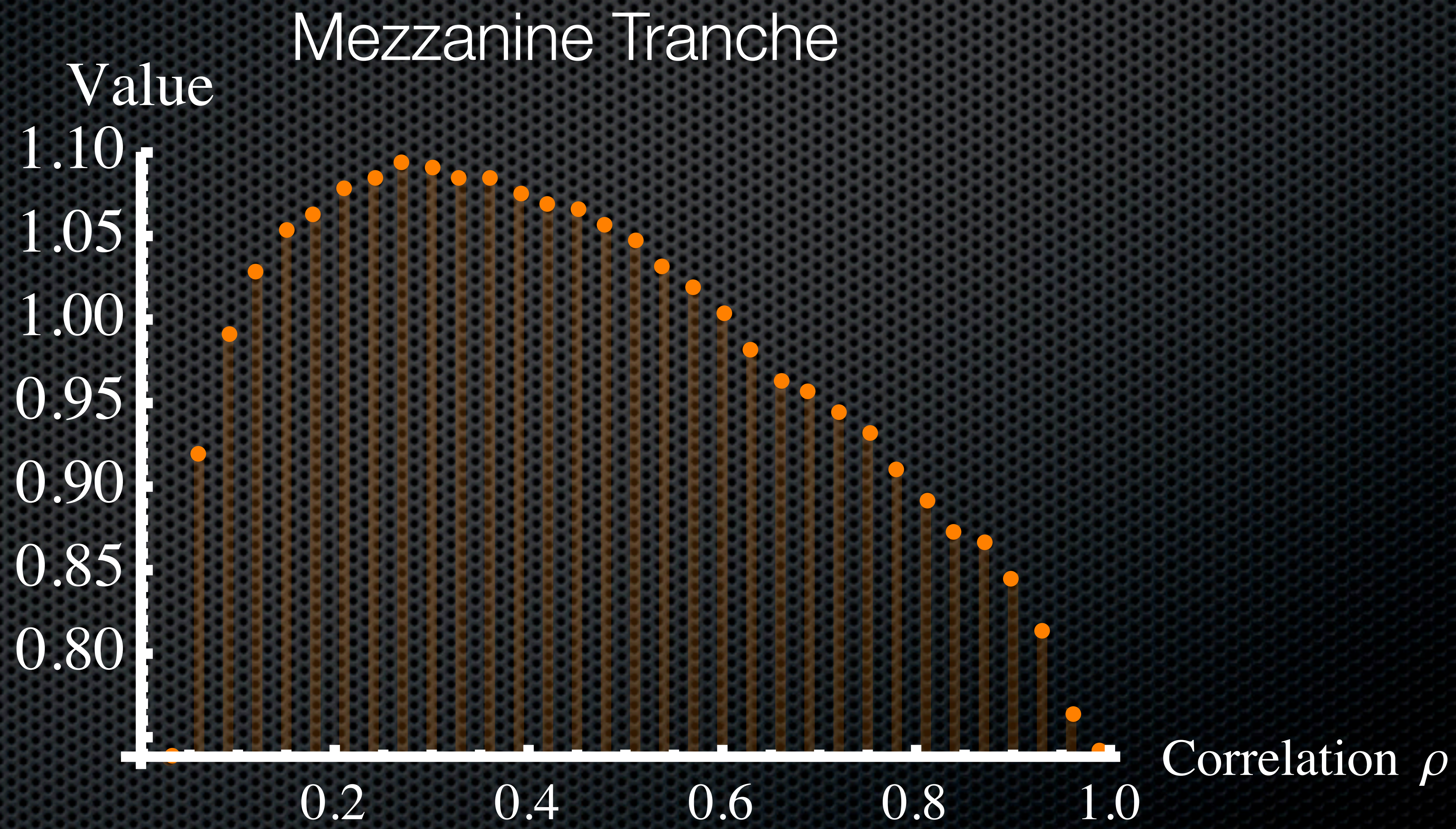
Correlation and Protection Value



Correlation and Protection Value



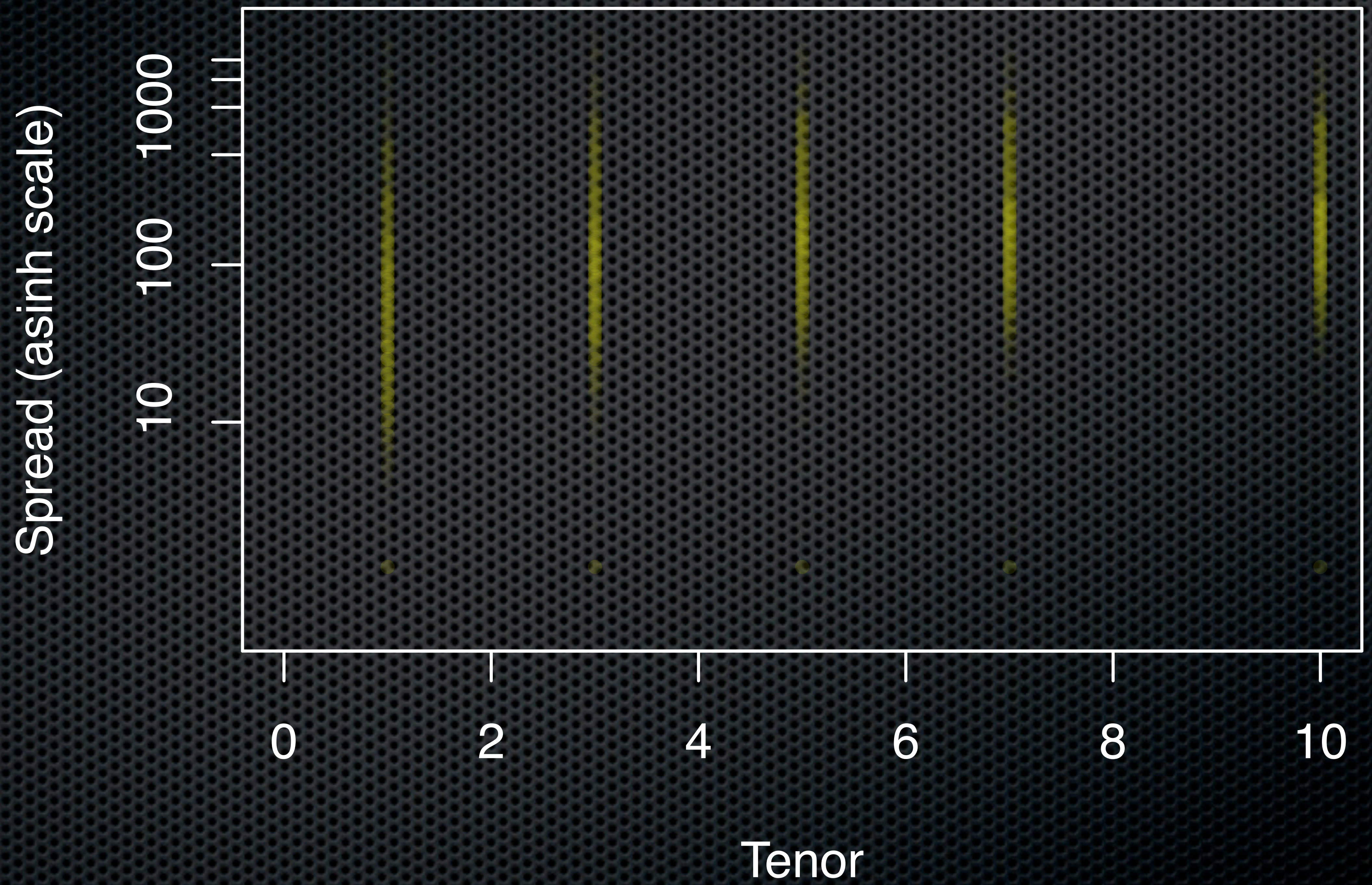
Correlation and Protection Value



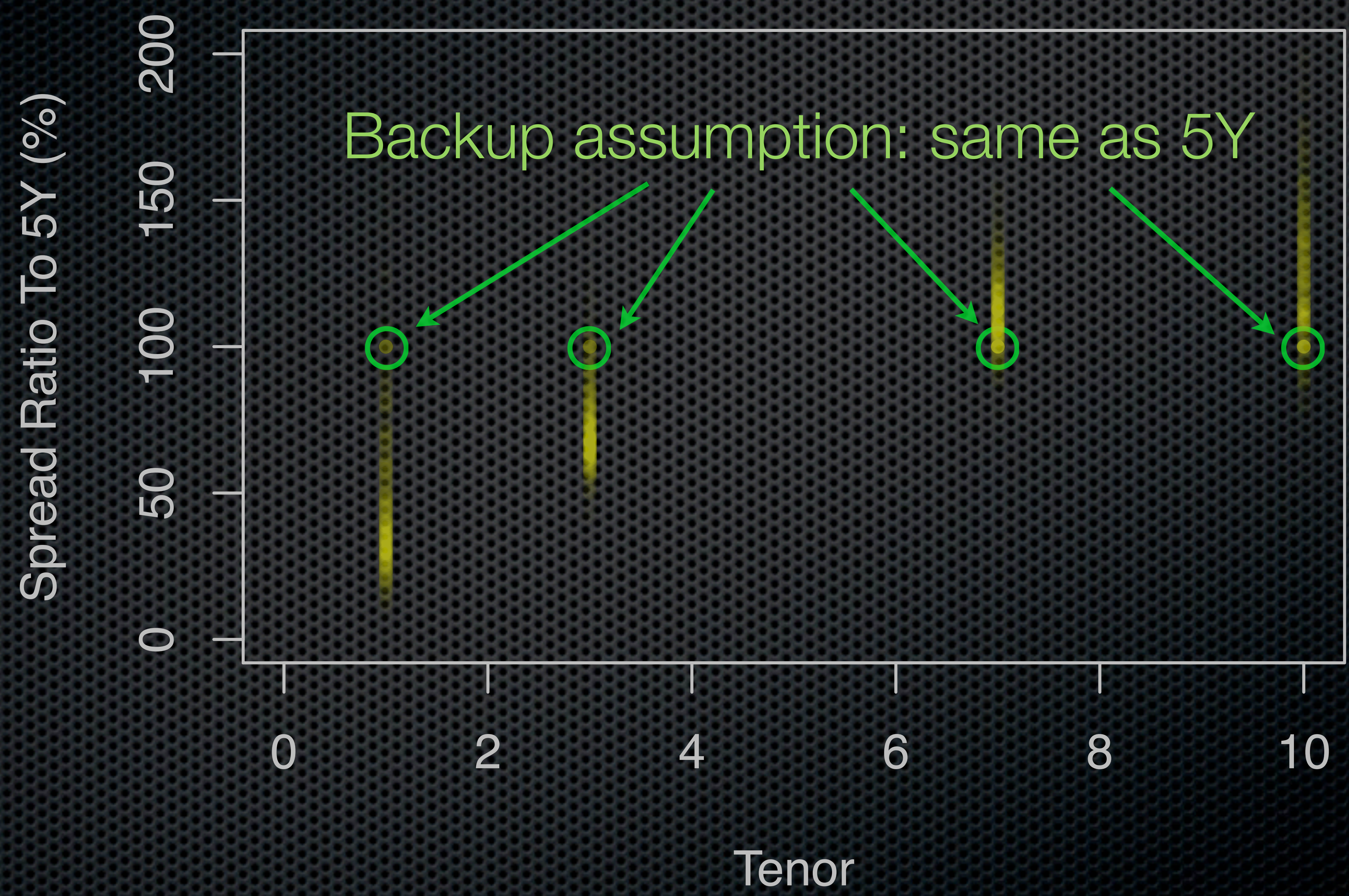
The Messy Word of Multivariate Credit

- Difficulties in single-name modeling
  - Fuzzy prices, unobserved events
  - Guessing at hazard rate and recovery
  - Risk curves on little data
- Flaws in copula models
  - Guessing at correlation
  - Skinny tails in multivariate gaussian
- How to model changes in credit instrument prices?

Risk Curves: US Market



Risk Curve Ratios: US Market



Default Times Under High Correlation

$$\tau_A = -\frac{1}{h_A} \log\left(N(z_A)\right)$$

$$\tau_B = -\frac{1}{h_B} \log\left(N(z_B)\right)$$

$$\langle z_A, z_B \rangle = \rho$$

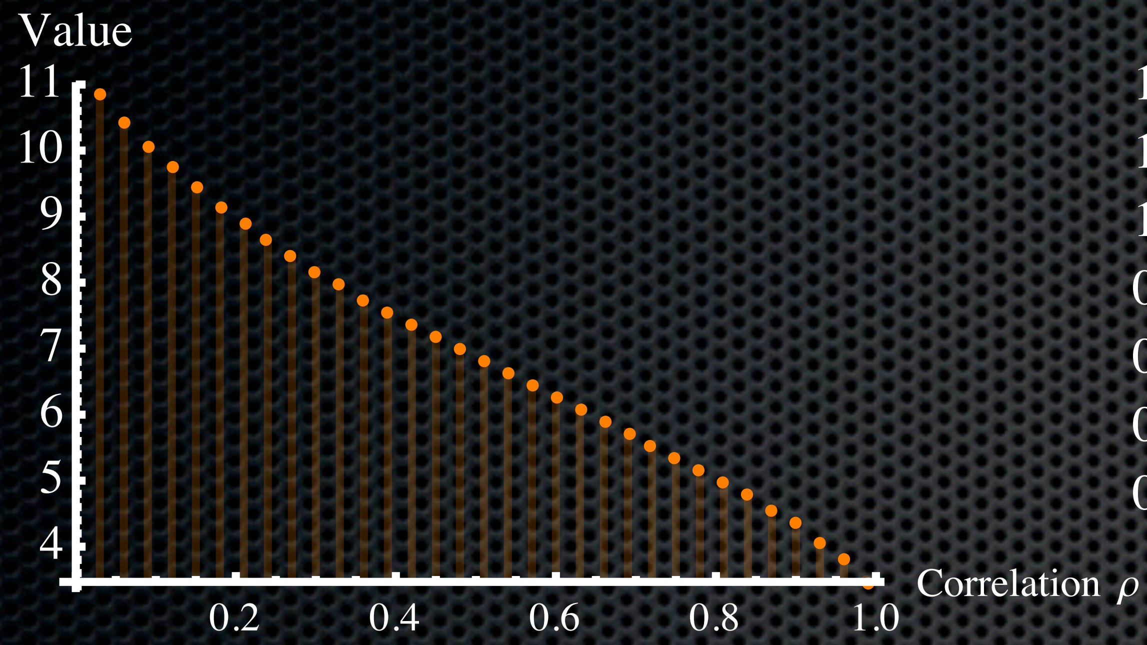
Default Times Under High Correlation

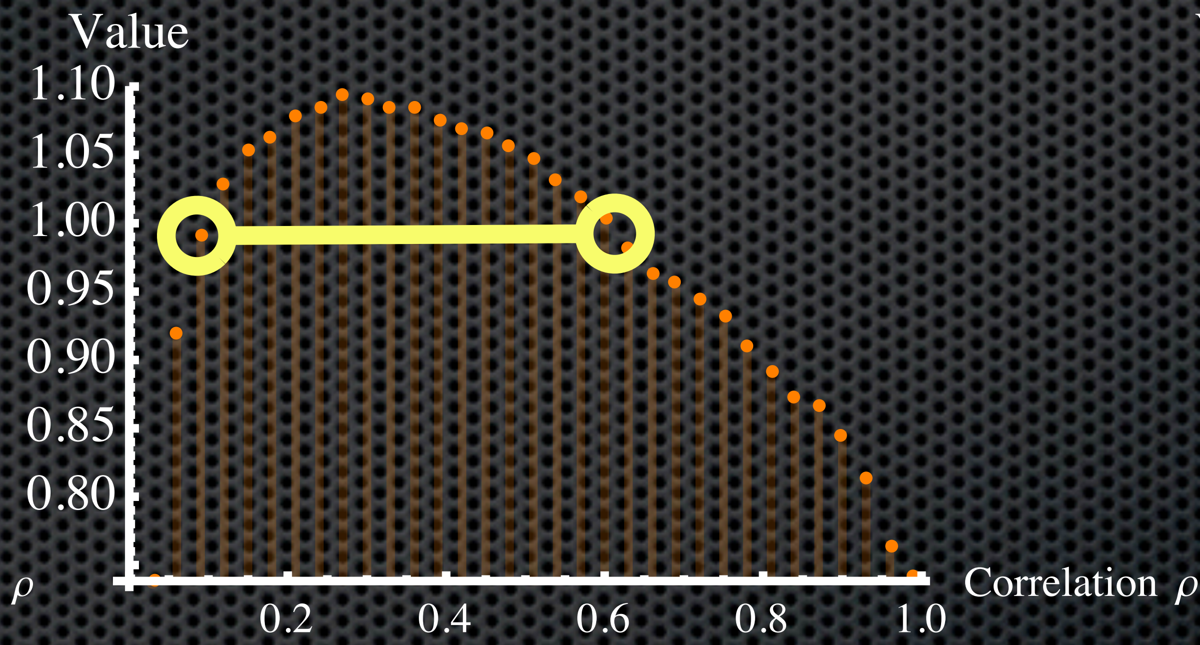
$$\begin{aligned}
\tau_B \big| \{\tau_A \equiv T\} &\sim -\frac{1}{h_B} \log \left(N \left(-\rho N^{-1} \left(1 - e^{-h_A T} \right) - w\sqrt{1-\rho^2} \right) \right) \\
&\approx -\frac{1}{h_B} \log \left(N \left(N^{-1} \left(e^{-h_A T} - 1 \right) \right) \right) \\
&= -\frac{1}{h_B} \log \left(e^{-h_A T} \right) \\
&= \frac{h_A}{h_B} T
\end{aligned}$$

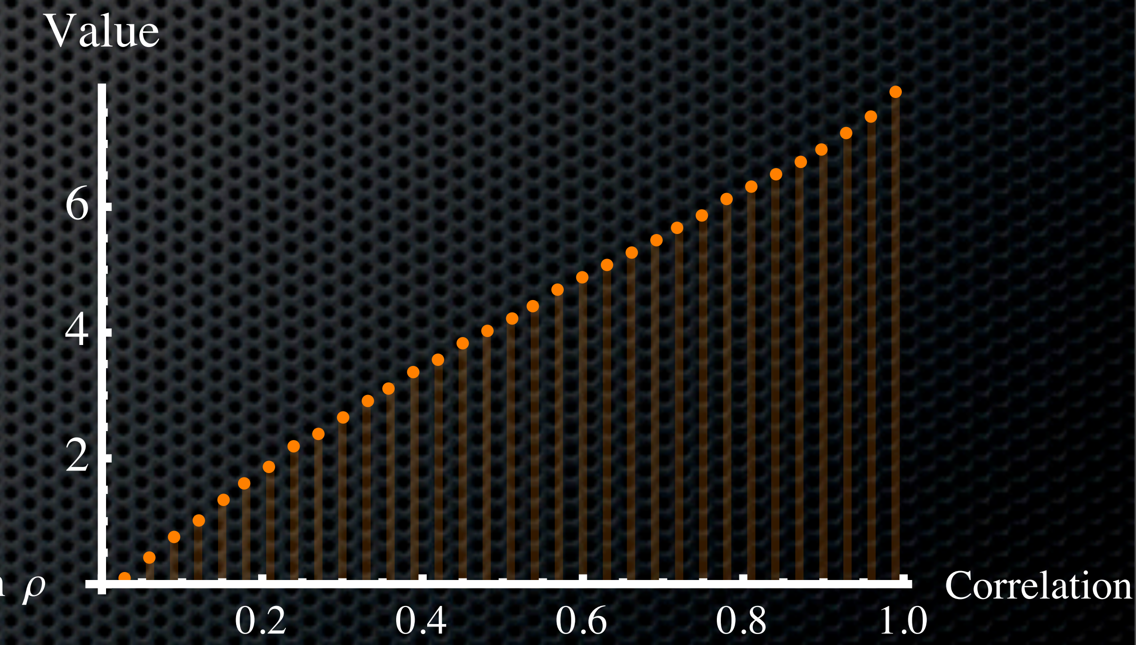
Default Times Under High Correlation

- Under high correlations, conditional default times are known almost exactly from h_A/h_B. Examples:
  - Conditional on $\tau_A = 1$ year, we know *B* must have defaulted at precisely 6 months
  - Conditional on $\tau_A = 1$ year, we know *B* will default at precisely 18 months
- Highly counterintuitive. On the other hand, such high correlations are not really credible

Correlation and Tranche Protection







Correlation and Tranche Protection

- Protection of the entire portfolio is independent of correlation
- Similar to forward price being independent of volatility
- Correlation of tranches has a “smile” much like options volatility has a skew

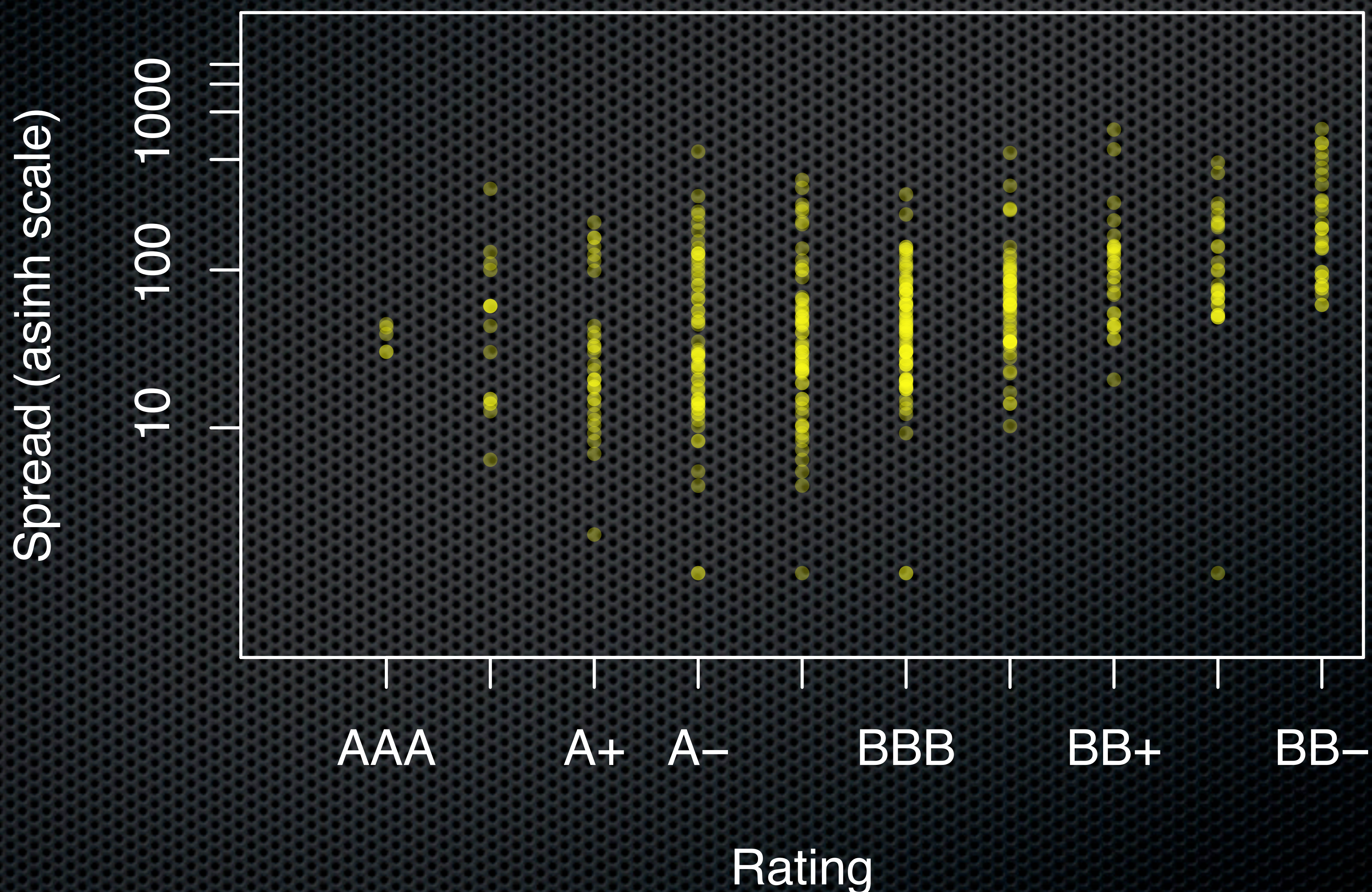
Base Correlation

- Protection of equity tranches is always monotonic in correlation
  - Treat mezzanine tranches as layers on top of equity tranches
  - Sum prices to get price of a “super-equity” tranche
  - Infer correlation
- Typically encounter a “smirk”

The Role of Ratings

- Ratings agencies: Moody's, Fitch, Standard & Poor's, others
- Used by market players and regulators to judge credit risk
- Typically concentrated on expected loss (EL)
    - Convolves loss-given-default (LGD) with loss probability
    - We want to separate them!
- Lagging indicators

Do Ratings Tell Us About Default Risk?



Tranche Protection and Ratings

- Ratings agencies had expertise in judging the risk of real businesses
- They started using models to assess portfolio risks
  - Binomial and copula
  - Ultimately applied to tranche protection
- Copied ratings labels from real businesses to synthetic portfolios
- Disastrous results

Ratings Migration Models

- Finite-state Markov models: a generalization of copulas
- All members of same rating assumed equivalent
- Transition from one rating to another with given probabilities
- Able to approximate PL by assigning value to ratings classes
- One year historical data: requires matrix logarithms and matrix roots for shorter time periods

Non-default Outcomes

- Credit *risk* can change even if a company does not default
  - Poor earnings, mergers, capital structure changes
  - Result is a *change* in theoretical and market value
    - Mark-to-market risk
    - Portfolio volatility
- We need a way to model changes in value, possibly via changes in h

Non-default Outcomes

- Tempting observation: credit spreads and hazard rates behave like stock prices
  - Never below zero
  - Higher values have higher standard deviations
- Problem: spreads are abstract concepts
  - Recovery variation interferes with stability
  - Jumps

Non-default Outcomes

- Spreads are not investable
  - Investable securities are bonds, loans, CDS
  - How should one think of drift?
  - Is the distribution lognormal for any cogent reason?
- A common concept is *total return*: the return experienced by an investor in the contract who reinvests all cashflows

Classical Reasons to Avoid Asset Price

- Pull to par
- Price boundaries
  - Ceilings for bonds
  - Floors for CDS upfronts
- Zero and negative upfronts make poor divisors

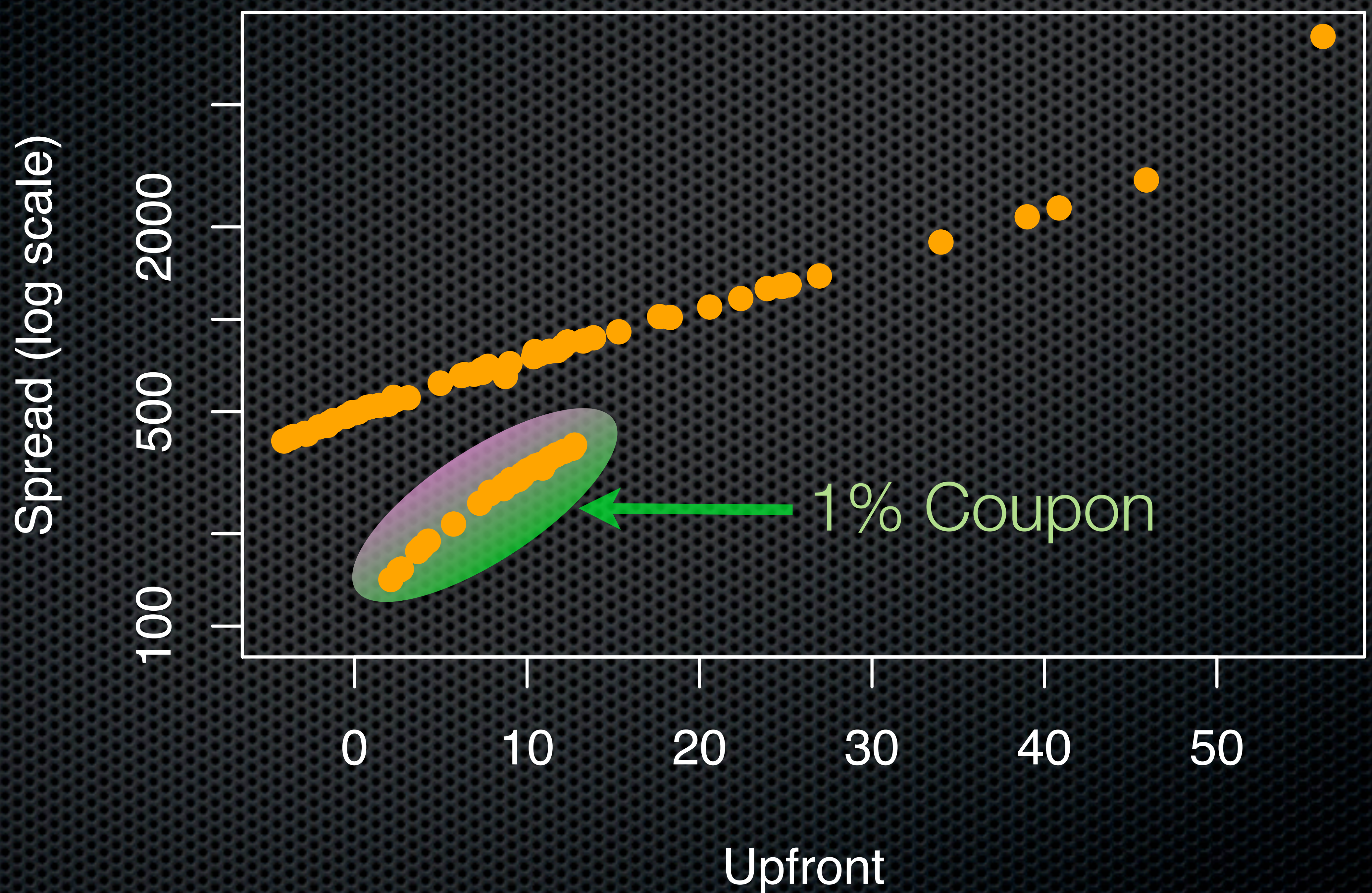
Non-default Outcomes

- We can consider total return or even Sharpe Ratio of a bond, loan, CDS or CDX investment
- We can compare total return series or asset prices to see how they compare to each other
- Common approach: consider credit instrument returns as variations on the liquidly traded **CDX**

Credit Indexes

- Indexes and associated tradable securities exist for most major credit categories
  - Europe, US, High Yield, Investment Grade, Industry groups
  - Play the same role as index futures and ETFs in equity world
- Credit is too quirky to define membership formulaically
  - Rebalancing by committee
  - Roll trading

CDX HY Spreads and Upfronts



Credit Index Example: CDX HY

Company	Spread	Upfront	Equity
First Data Corp	1014	18.30	
Goodyear Tire & Rubber Co	785	10.45	10.65
Dean Foods Co	557	2.26	11.71
Hertz Corp	468	-1.22	
Ford Motor Co	301	9.07	11.92
CMS Energy Corp	180	3.88	21.94

¼ of names have no equity

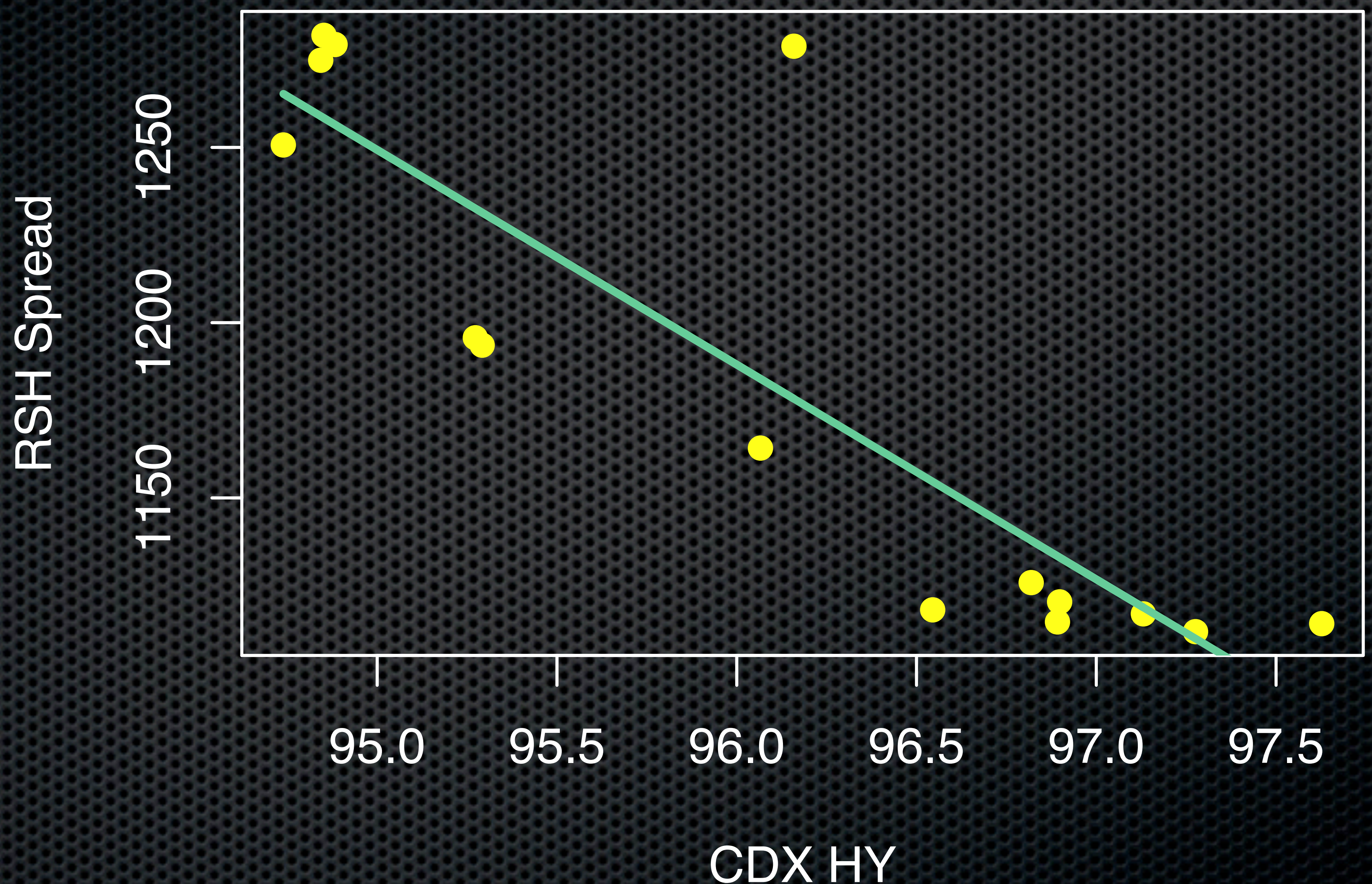
CDX and Radio Shack

Date	CDX	Spread	Upfront
3/27/12	97.627	1114.143	6.48
3/28/12	97.276	1111.866	6.43
3/29/12	96.819	1125.855	6.31
3/30/12	96.892	1114.626	6.22
4/2/12	97.131	1116.926	6.3
4/3/12	96.898	1120.352	6.15
4/4/12	96.545	1118.075	6.27
4/5/12	96.066	1164.233	6.05
4/6/12	95.292	1193.588	6.06
4/9/12	95.273	1195.707	
4/10/12	94.739	1250.707	5.84
4/11/12	94.852	1281.995	6.02
4/12/12	96.159	1278.883	6.11
4/13/12	94.843	1274.879	5.95
4/16/12	94.883	1279.368	5.99

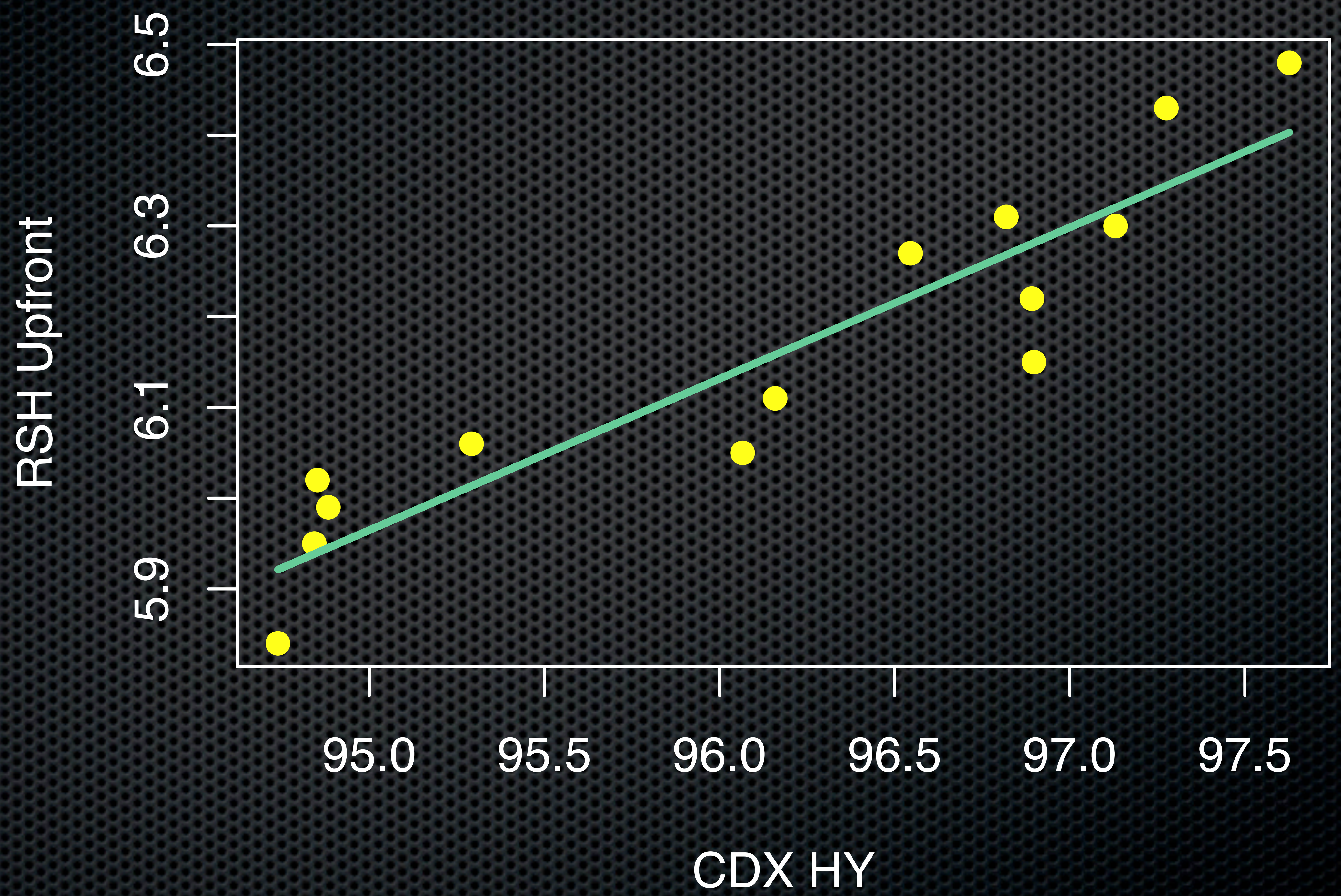
Linear Models For Credit

- An important way of using CDX to tell us about single names
- Essentially *all* models are locally linear
- We seek robustness, so we concentrate on linear models
- Important but arbitrary choices
    - Weighting and time periods
    - Choice of variable (spread, upfront, bond equivalent, total return)

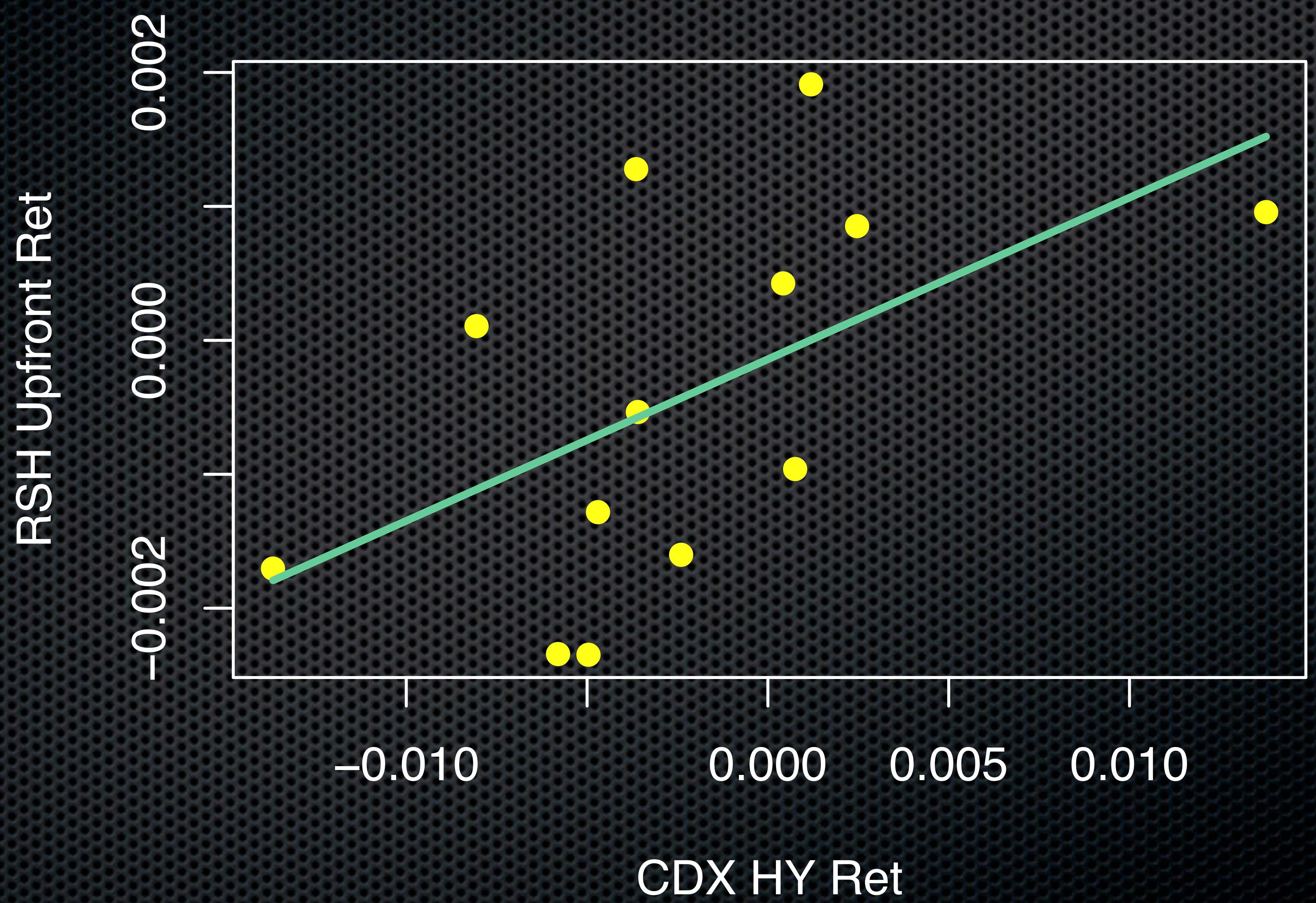
CDX and Radio Shack



CDX and Radio Shack



CDX and Radio Shack



Regressions Amplified

- We do not need to assume just one driver of values
- A multivariate regression allows us to assume asset prices are driven by multiple *factors*
- Simple linear algebra can be combined with empirical factor distributions

Factor Models

$$r_i = \alpha_i + \beta_{i,1} f_1 + \beta_{i,2} f_2 + \cdots + \beta_{i,N} f_N + e_i$$

Key Assumption:

$$\langle e_i, e_j \rangle = 0 \quad \text{for } i \neq j$$

Factor Model Advantages

- Huge reduction in dimensionality of parameter space
  - 1000 securities ⇒ ~500,000 covariances
  - 30 factors ⇒ 870 covariances + 30,000 β + 1,000 residuals
- PL Explanatories
- Intuitive choice of factors

More Information

- http://dtcc.com/products/derivserv/data_table_i.php
- http://www.markit.com/en/products/data/indices/credit-and-loan-indices/cdx/cdx.page?
- http://defaultrisk.com/
- **Counterparty Credit Risk** by John Gregory
- **Credit Derivatives Pricing Models** by Philipp Schönbucher

http://public.boonstra.org/MFCredit2012SlidesFinal.pdf